First Order Differential Eqns (Ch 9)

Three main equations:

1. Exponential Growth / Decay
2. Learning Curve
3. Logistic equation

amount present proportional to rate of change

1. $\frac{dy}{dt} = ky \rightsquigarrow \frac{dy}{y} = k\,dt$

$\int \frac{dy}{y} = \ln y = \int k\,dt = kt + C$

$e^{\ln y} = e^{kt+C}$

"initial value"

$y = e^{kt} \cdot e^{c}$ $\quad y(0) = A$

$A = (1) \cdot e^{c}$

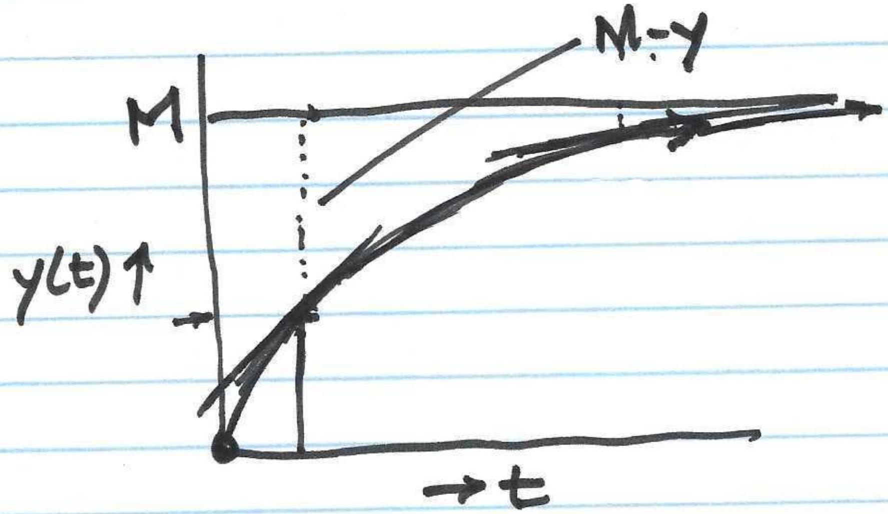
"growth/decay" constant

Final sol'n is $y(t) = Ae^{kt}$ $\quad \lambda > 0$

$P(t) = P(0)e^{kt} \mid Q(t) = Q(0)e^{-\lambda t}$

(2) Learning Curve $y(t)$ is amount

$$\frac{dy}{dt} = k(M-y)$$



$$\frac{dy}{M-y} = k\,dt$$

$$\int \frac{dy}{M-y} \overset{u=M-y}{=} \int \frac{-du}{u} = -\ln|u| = -\ln|M-y|$$

$$\int k\,dt = kt + C$$

$$e^{-\ln(M-y)} = e^{kt+C} \Rightarrow$$

$$e^{\ln\left(\frac{1}{M-y}\right)} = \frac{1}{M-y} = e^{kt+c}$$

$$M-y = e^{-kt}\cdot e^{-c}$$

$$\boxed{M - e^{-kt}\cdot e^{-c}} = y(t)$$

$$y(0) = 0$$

$$M - Ae^{-kt} = y(t)$$

$$\underline{M - A} = 0 \Rightarrow A = M$$

$$M - Me^{-kt} = y(t)$$

$$M(1 - e^{-kt}) = y(t)$$

Applies to acquisition of knowledge or acquisition of "news" that is broadcast to a fixed population.

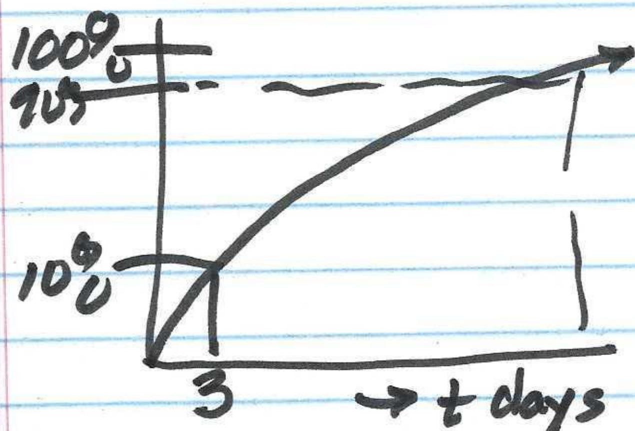
If a group of citizens (1000 people) hear on radio that a bad storm is

$$\ln(.25) = -.287\,t$$

$$-1.386 = -.287\,t$$

So $t = \dfrac{1.386}{.287} = \underline{4.829 \text{ hrs}}$

A play is in rehearsal. The star knows 10% of his lines after 3 days. A dress rehearsal is to be scheduled when he knows 90% of his script. How long after he started memorizing will this be.



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Learning Curve Model: $y(t) = \overset{100\%}{M}(1-e^{-kt})$

Data point: $y(3) = .10$

$$y(3) = .10 = (1)(1-e^{-k(3)})$$

$$.10 = 1-e^{-3k}$$

$$e^{-3k} = .90$$

$$-3k = \ln(.90) = -.105$$

$$\Rightarrow k = \frac{.105}{3} = .035$$

Reassemble model:

$$y(t) = (1)(1-e^{-.035t})$$

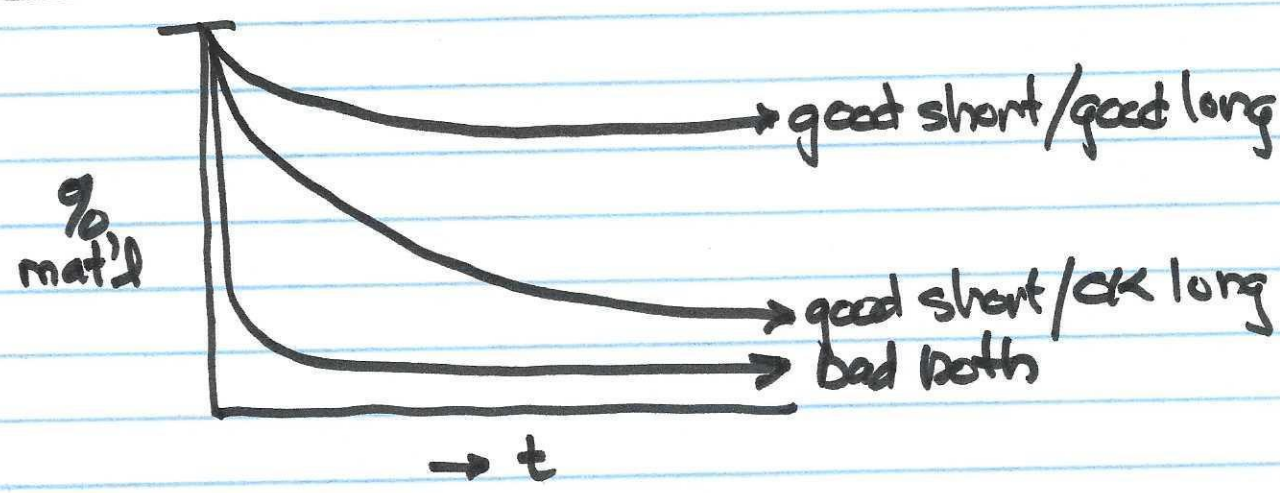
$$.90 = 1-e^{-.035t}$$

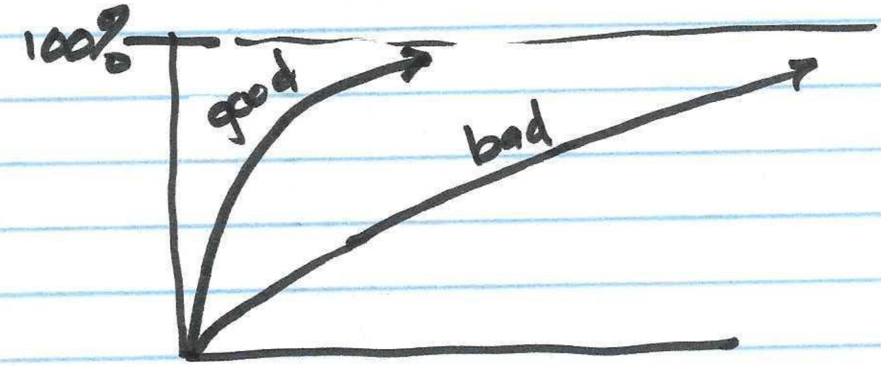
$$e^{-.035t} = .10$$

$$-.035t = \ln(.10) = -2.3$$

$$t = \frac{2.3}{.035} = 66$$

Ebbinghaus





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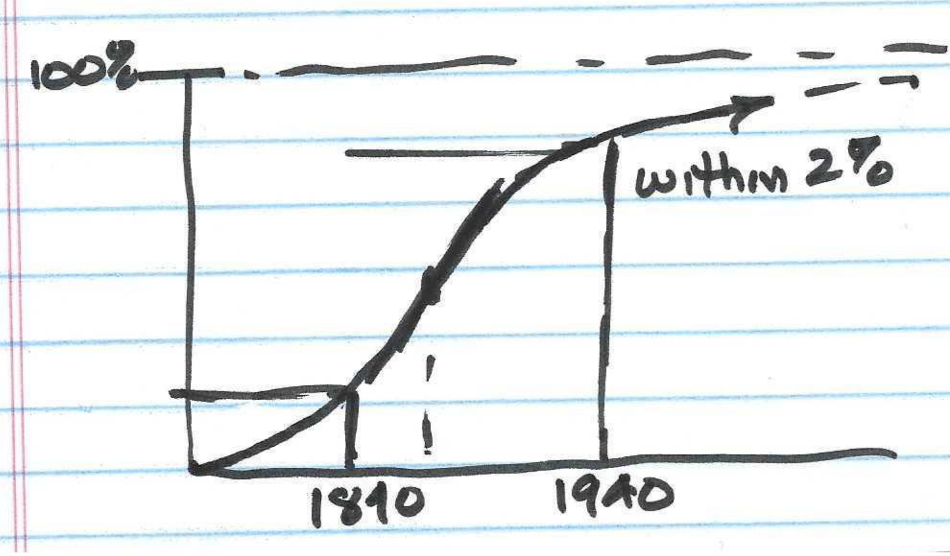
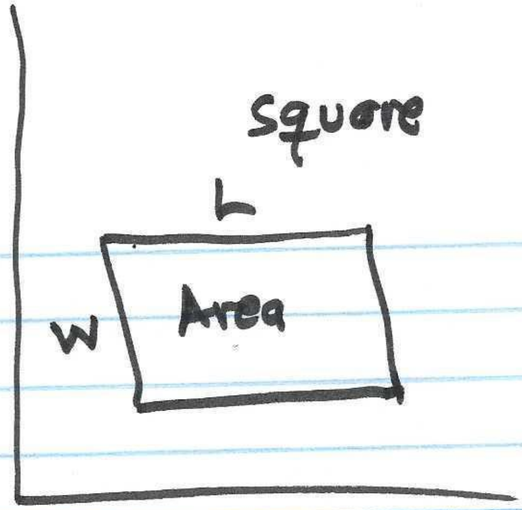
Logistic Curve / ver Hulst Equation





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3 $\frac{dy}{dt} = ky \cdot (M - y)$

$$\frac{dy}{y(M-y)} = k\,dt$$

$$\int \frac{dy}{y(M-y)} = kt + C$$

↑ decompose with partial fractions

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y} = \frac{1}{y} + \frac{1}{M-y}$$

$$\frac{\cancel{y} + 1 - \cancel{y}}{y(1-y)}$$

$1 = A(M-y) + By$

$1 = AM - Ay + By$

$-A + B = 0 \Rightarrow A = B$

$1 = AM$ say $M = 100\%$ or 1

$A = 1$

$$\int \frac{dy}{y(1-y)} = \int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln|y| + \ln|1-y|$$

Summary: $\ln y + \ln(1-y) = kt + C$

$$\ln(y(1-y)) = kt + C$$

$$y(1-y) = e^{kt} \cdot e^{C} \qquad y(0) = A$$

$$A(1-A) = e^{C}$$

$$y(1-y) = A(1-A)e^{kt}$$

$$\underline{y(t) = \frac{M}{1 + Be^{-kMt}}}$$

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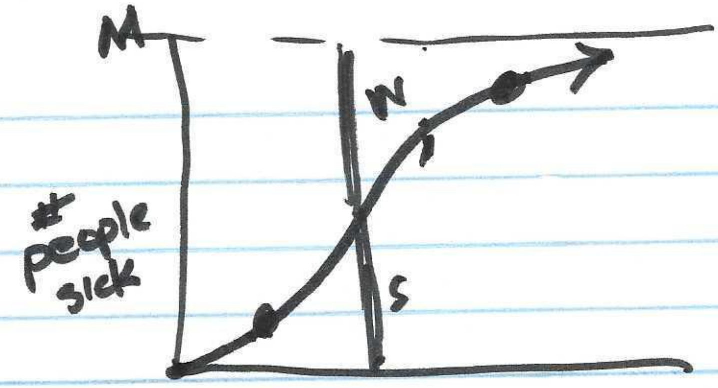

$M = 1000$

t in days

@ $t = 0$ 10 people

@ $t = 5$ 25 people

Question: When will half people be sick?

$$y(0) = \frac{1000}{1+Be^{0}} = 10$$

$$1000 = 10(1+B)$$

$$10B = 990 \Rightarrow \boxed{B = 99}$$

$$y(t) = \frac{1000}{1+99e^{-kMt}}$$

$$y(5) = \frac{1000}{1+99e^{-k(5000)}} = 25$$

$$25\left(1+99e^{-5000k}\right) = 1000$$

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$$1+99e^{-5000k}=40$$

$$99e^{-5000k}=39$$

$$\underline{e^{-5000k}}=\frac{39}{99}=\underline{.39}$$

$$-5000k=\ln(.39)=-0.942$$

$$k=\frac{0.942}{5000}=\frac{.000188}{.188}$$

$$\boxed{y(t)=\frac{1000}{1+99e^{-.188t}}}$$

$$500=\frac{1000}{1+99e^{-.188t}}$$

$$500(1+99e^{-.188t})=1000$$

$$1+99e^{-.188t}=2$$

$$t=\frac{4.6}{.188}=\underline{25\text{ days}}$$

$$99e^{-.188t}=1$$

$$e^{-.188t}=\frac{1}{99}\quad 2\ \ln?$$

$$+.188t=+4.6$$