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Improper (Infinite) Integrals

Three kinds:

Type I: one or both limits of integration are ∞

Type II: $f(x) = \infty$ at some point in the interval of integration

Type III: Both Type I & Type II happen

Ex. (Type I): $\int_0^\infty e^{-x}\,dx = 1$

means $\lim_{M\to\infty} \int_0^M e^{-x}\,dx$

$\left[-e^{-x}\right]_0^M = -e^{-M} - (-e^{-0}) =$

$1 - e^{-M}$ $\frac{1}{e^M}$

$\lim_{M\to\infty} (1 - \underset{\to 0}{e^{-M}}) \overset{?}{=} 1$

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Ex. $\displaystyle\int_1^{\infty} \frac{dx}{x} = \lim_{M\to\infty} \int_1^{M} \frac{dx}{x} = \Big[\ln x\Big]_1^M = 2$

$\ln M - \underbrace{\ln 1}_{0} = \ln M$

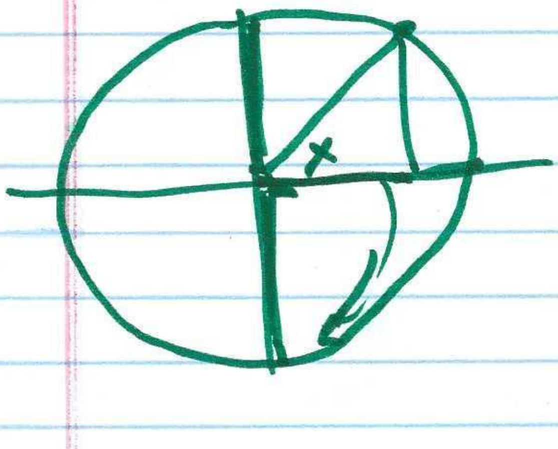
$\displaystyle\lim_{M\to\infty} \ln M = \infty$

So $\displaystyle\int_1^{\infty} \frac{dx}{x} = \infty$ ← when this occurs we say the integral

divergesy

Ex. $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^2+1} = ?$ $\Big[\arctan x\Big]_{-\infty}^{0} = 0 - \left(-\frac{\pi}{2}\right)$

$\boxed{= \frac{\pi}{2}}$



$\arctan 1 = \frac{\pi}{4}$

$\arctan 0 = 0$

$\arctan +\infty = \frac{\pi}{2}$

$\arctan -\infty = -\frac{\pi}{2}$

Ex. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx \stackrel{?}{=}$

$\frac{1}{x}$

$$\int \frac{\ln x}{x}\, d(\ln x)$$

$$\int \frac{\ln x \, d(\ln x)}{x} \stackrel{?}{=}$$

Let $u = \ln x$

$dv = \dfrac{dx}{x^2}$

then $du = \dfrac{dx}{x}$

$v = -\dfrac{1}{x}$

$$\int \underbrace{\ln x}_{u}\, \underbrace{\frac{dx}{x^2}}_{dv} = \ln x\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{dx}{x}$$

$$= -\frac{\ln x}{x} + \int \frac{dx}{x^2} = \boxed{-\frac{\ln x}{x} - \frac{1}{x}}$$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{x\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right] + 1.$$

What is $\lim_{x\to\infty}\left[-\frac{\ln x}{x} - \cancel{\frac{1}{x}}\right]$

$\frac{\ln x}{x} \to 0 \qquad \frac{1}{x} \to 0$

L'Hôpital's Rule

$$\lim_{x\to\infty}\left(-\frac{\ln x}{x}\right) = \lim_{x\to\infty}\left(-\frac{1/x}{1}\right) \to 0$$

Ex $\int_1^\infty \frac{dx}{x^p} \qquad p \in \mathbb{R}$

$\int x^{-p}\,dx$

$\frac{x^{-p+1}}{-p+1}$

$p = 1 \qquad \left[\ln x\right]_1^\infty = \infty$

$p > 1 \qquad \left[\frac{x^{1-p}}{1-p}\right]_1^\infty$

$1-p = -2$

$\frac{1}{x^2}$

$$\left[0 - \frac{1}{1-p}\right] = \frac{1}{p-1}$$

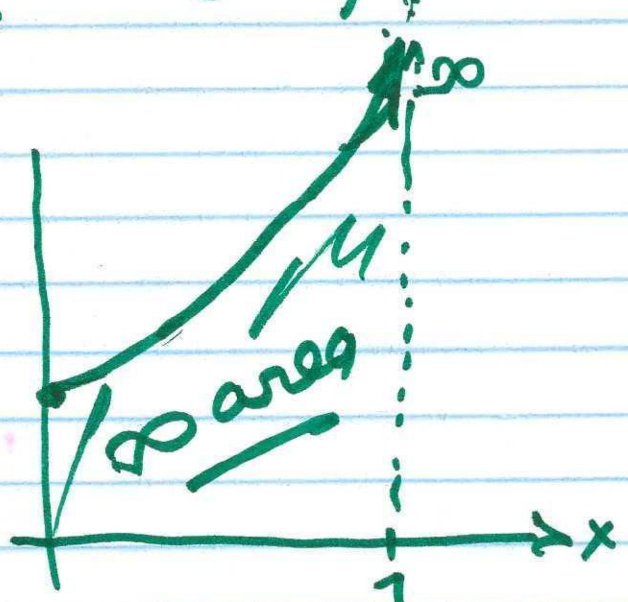
Ex. $$\int_0^1 \frac{dx}{1-x} = \lim_{\alpha \to 1} \int_0^\alpha \frac{dx}{1-x}$$

$$u = 1-x \Rightarrow du = -dx$$

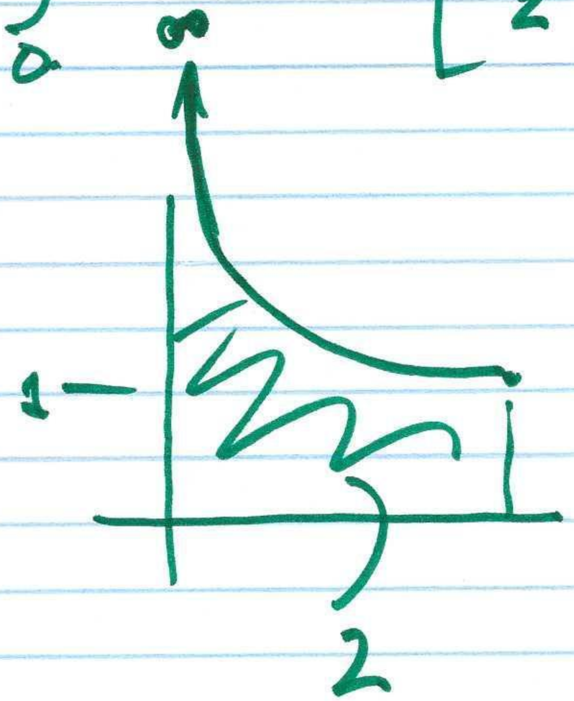
$$\int_1^{1-\alpha} \frac{1}{u}(-du) = \Big[-\ln u\Big]_1^{1-\alpha}$$

$$\underline{-\ln(1-\alpha)} - (-\ln 1)$$

$$\lim_{\alpha \to 1}\big(-\ln(1-\alpha)\big) = ? \rightsquigarrow \underline{-\ln 0} = +\infty$$



$$\int_0^1 \frac{dx}{\sqrt{x}} \stackrel{?}{=} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1 = \left[2\sqrt{x}\right]_0^1 = 2$$



Convergence Testing for Integrals

$$\int_1^\infty \frac{\sin^2 x \, dx}{x^2} \stackrel{?}{=}$$

$f(x) \le g(x)$

$$\int_1^\infty \overbrace{\frac{\sin^2 x}{x^2}}^{f(x)} dx \le \int_1^\infty \overbrace{\frac{dx}{x^2}}^{g(x)} = \left[-\frac{1}{3} \cdot \frac{1}{x^3}\right]_1^\infty = +\frac{1}{3}$$

Theorem Let $f(x)$ & $g(x)$ be continuous on the interval $[a, \infty)$. Further suppose that $0 \le f(x) \le g(x)$ for all $x \ge a$.

1. $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

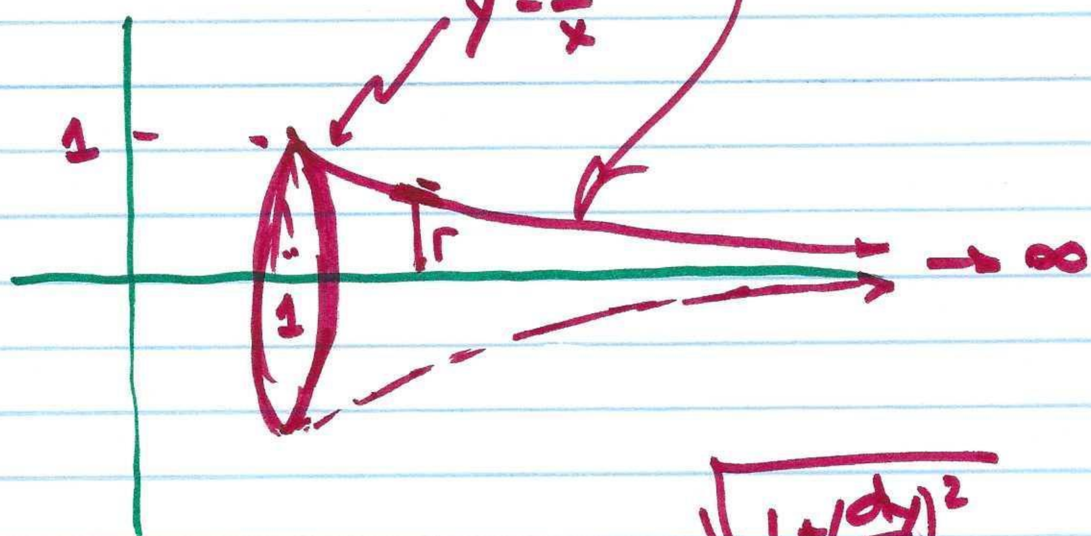
2. $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges

Theorem If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = L$ $\quad \underline{0 < L < \infty}$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either

both converge or diverge.

82 "Gabriel's Horn"



$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

What is surface area?

$$\text{Surface area} = \int_1^{\infty} 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}}\, dx$$

$$\text{Volume} = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx$$

(9)

$$V = \pi \int_1^\infty \frac{dx}{x^2} = \pi\left(\frac{-1}{3x^3}\right)\Bigg|_1^\infty = \pi \cdot \frac{1}{3} = \boxed{\frac{\pi}{3}}$$

finite volume

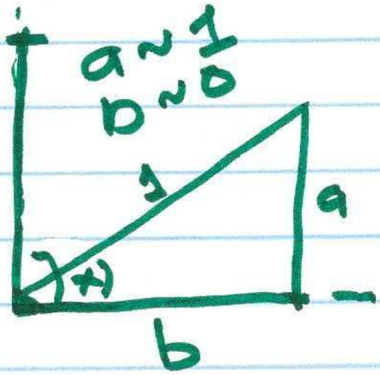
We know $\displaystyle\int_1^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}}\,dx \geq \int_1^\infty \frac{2\pi}{x}\,dx$

What is $\displaystyle\int_1^\infty \frac{2\pi\,dx}{x} = 2\pi\Big[\ln x\Big]_1^\infty = \infty$

so... $\displaystyle 2\pi\int_1^\infty \frac{\sqrt{1+\frac{1}{x^4}}}{x}\,dx \rightsquigarrow \infty$

infinite area

1. $\int_0^\infty \frac{dx}{x^2+1} = ?$ $\quad \left[\arctan x\right]_0^\infty = \frac{\pi}{2} - 0$



$\tan x = a/b = \infty$

$\arctan \infty = \frac{\pi}{2}$

2. $\int_1^\infty \frac{dx}{x^{1.001}}$ $\quad x^{-1.001} \rightsquigarrow \frac{x^{-.001}}{-.001} =$

$-\left[1000x^{-.001}\right]_1^\infty = \boxed{1000}$

$\frac{1}{x^{.001}}\Big|^\infty = 0$

6 $$\int_{-8}^{1} \frac{dx}{\sqrt[3]{x}} \overset{?}{=} \int_{-8}^{0} \frac{dx}{x^{1/3}} + \int_{0}^{1} \frac{dx}{x^{1/3}}$$

$$\left[\frac{3}{2}x^{2/3}\right]_{-8}^{0} + \left[\frac{3}{2}x^{2/3}\right]_{0}^{1}$$

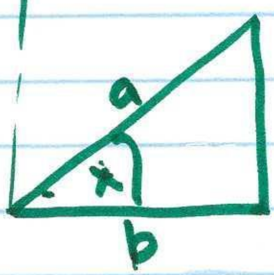
$$\frac{3}{2}(0-4) + \frac{3}{2}(1-0)$$

$$\frac{3}{2}(-3) = \boxed{-\frac{9}{2}}$$

7 $$\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} \overset{?}{=} \int_{0}^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \frac{\pi}{2}$$

Let $x = \sin\theta$ $\quad dx = \cos\theta\, d\theta$

18 $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \Big[\operatorname{arcsec}|x|\Big]_1^\infty = \boxed{\frac{\pi}{2}-0}$



$\sec x = \frac{a}{b}$

$x = \operatorname{arcsec}\left(\frac{a}{b}\right)$

$\operatorname{arcsec} 1 = 0$

21 $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\, d\theta$

$\displaystyle\int \theta e^{\theta} d\theta = \theta e^{\theta} - \int e^{\theta} d\theta = \underline{\underline{\theta e^{\theta} - e^{\theta}}}$

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$$\left[\theta e^{\theta}-e^{\theta}\right]_{-\infty}^{0}=(-1)-(0)=\boxed{-1}$$