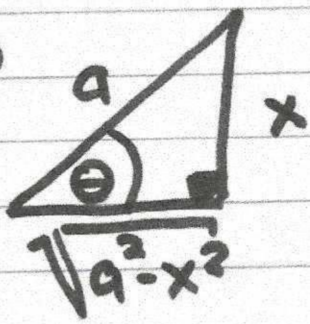
MAGIC TRIANGLES

To handle $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, and $\sqrt{x^2+a^2} = \sqrt{a^2+x^2}$ in integrals

(i)

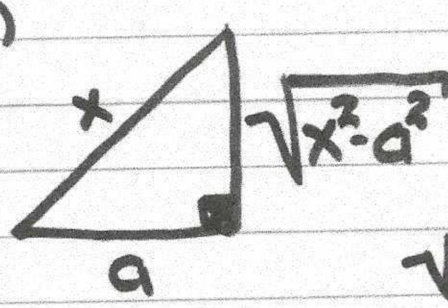


Use $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

(ii)

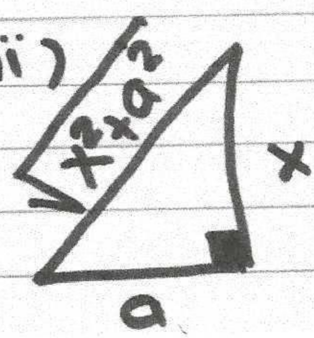


Use $x = a\sec\theta$

$dx = a\sec\theta\tan\theta \, d\theta$

$\sqrt{x^2-a^2} = a\tan\theta$

(iii)



Use $x = a\tan\theta$

$dx = a\sec^2\theta \, d\theta$

$\sqrt{x^2+a^2} = a\sec\theta$

Ex. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$

$\frac{x^2}{4} = \sec^2\theta - 1$

$\sec\theta = \sqrt{1+\frac{x^2}{4}}$

Use $x = a\tan\theta = 2\tan\theta$

$dx = a\sec^2\theta \, d\theta$

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4}$

$2\sqrt{\tan^2\theta+1} = 2\sec\theta$

$$\int \frac{\cancel{2}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{2}\cancel{\sec\theta}} = \int \sec\theta \, d\theta$$

$$\ln|\sec\theta + \tan\theta| + C$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \ln\left|\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right| + C$$

$$= \ln\left|\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right| + C$$

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$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

Set $x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$\int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,(3\cos\theta\,d\theta)$$

$$\frac{27}{3}\int \frac{\sin^2\theta\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = 9\int \sin^2\theta\,d\theta$$

$$\frac{9}{2}\int (1-\cos 2\theta)\,d\theta =$$

$$\frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

3

Exercise Set 8.1

8

$$\int \sqrt{1-9t^2}\,dt \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$t = \frac{1}{3}\sin\theta$$

$$\sqrt{1-9t^2} \rightsquigarrow \sqrt{1-\sin^2\theta}$$

$$dt = \frac{1}{3}\cos\theta\,d\theta$$

$$\int \underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta}\left(\frac{1}{3}\cos\theta\,d\theta\right)$$

$$\frac{1}{3}\int \cos^2\theta\,d\theta = \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$\frac{1}{6}\int (1+\cos 2\theta)\,d\theta$$

4

$$\frac{1}{6}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

12 $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy \qquad y>5$

Let $y = 5\sec\theta$

$dy = 5\sec\theta\tan\theta\,d\theta$

$$= 5\int \frac{\sqrt{25\sec^2\theta - 25}}{125\sec^3\theta}\sec\theta\tan\theta\,d\theta$$

$$= \frac{5\cdot 5}{5^3}\int \frac{\sqrt{\sec^2\theta - 1}}{\sec^3\theta}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{5}\int \frac{\tan\theta}{\sec^3\theta}\sec\theta\tan\theta\,d\theta$$

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$$\frac{1}{5}\int \frac{\tan^2\theta\, d\theta}{\sec^2\theta} = \frac{1}{5}\int \frac{\sec^2\theta - 1}{\sec^2\theta}\, d\theta$$

$$\frac{1}{5}\int \left(1 - \frac{1}{\sec^2\theta}\right) d\theta = \frac{1}{5}\int (1 - \cos^2\theta)\, d\theta$$

$$= \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{5}\int \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$

keep going

Alternate

$$\frac{1}{5}\int \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \cancel{\cos^2\theta}\, d\theta$$

35

$$\int_0^{\ln 4} \frac{e^t\, dt}{\sqrt{e^{2t}+9}}$$ Let $x = e^t$

$dx = e^t\, dt$

$$\int_{t=0}^{\ln 4} \frac{dx}{\sqrt{x^2+9}}$$ Now let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

⑥ $\sqrt{9\tan^2\theta+9}$

$3\underline{\sqrt{\tan^2\theta+1}}$

$$\int_{t=0}^{\ln 4} \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

$$\int_{t=0}^{\ln 4} \sec\theta\, d\theta = \ln\left|\sec\theta+\tan\theta\right|$$

Note: $\frac{x}{3} = \tan\theta$ so $\arctan\left(\frac{x}{3}\right) = \theta$

$t=0 \rightarrow e^{t} = 1 = x_{low}$

$t=\ln 4 \rightarrow e^{\ln 4} = 4 = x_{upper}$

$$\int_{0}^{\ln 4} t \rightarrow \int_{1}^{4} x \longrightarrow \int_{\arctan 1/3}^{\cancel{4/3 \arctan 4/3}\ \arctan 4/3} \theta$$

$$\Big[\ln\left|\sec\theta+\tan\theta\right|\Big]_{\arctan 1/3}^{\arctan 4/3} =$$

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$$\int \sqrt{8-2x-x^2}\,dx$$

$$-(x^2+2x-8)$$

$$-((x+1)^2-9)$$

$$\int \sqrt{-(x+1)^2+9}\,dx \quad \text{or} \quad \int \sqrt{9-y^2}\,dy$$

where $y = x+1$
$dy = dx$

Let $y = 3\sin\theta$, $dy = 3\cos\theta\, d\theta$

$$\int \sqrt{9-9\sin^2\theta}\,(3\cos\theta\, d\theta)$$

$$= \int 3^2\cos^2\theta\, d\theta = 9\int \cos^2\theta\, d\theta$$

$$= 9\int \frac{1+\cos 2\theta}{2}\, d\theta$$

⑧

$$\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+4}}$$

Now set $y = x-1$, then $dy = dx$

$$\int \frac{dy}{\sqrt{y^2+4}}$$

Let $y = 2\tan\theta$ $\quad dy = 2\sec^2\theta\,d\theta$

$$\int \frac{2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}} \rightarrow \int \frac{\sec^2\theta\,d\theta}{\sqrt{\tan^2\theta+1}}$$

$$\rightarrow \int \frac{\sec^2\theta\,d\theta}{\sec\theta} = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$