Partial Fraction Integration

Method to handle integrals of rational functions, i.e.

$$\int \frac{f(x)}{g(x)}\,dx$$

where $f(x), g(x)$ are polynomials.

Ex.: $$\int \frac{5x-3}{x^2-2x-3}\,dx = ?$$

$$= \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

same

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)+3(x+1)}{(x+1)(x-3)}$$

$$= \frac{5x-3}{x^2-2x-3}$$

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$$\int \frac{2dx}{x+1} + \int \frac{3dx}{x-3}$$

Let $u = x+1$

$du = dx$

$$2\int \frac{du}{u} \qquad \int \frac{3\,du}{u}$$

$$2\ln|u| \qquad 3\ln|u|$$

$$\boxed{2\ln|x+1| + 3\ln|x-3| + C = \int \frac{5x-3}{x^2-2x-3}\,dx}$$

Four things can happen when factoring the denominator:

1. Linear unrepeated factors
2. Some linear factors repeated
3. Quadratic unrepeated factors
4. Some quadratic factors repeated

1 Linear Unrepeated

$$\frac{5x-3}{x^2-2x-3} = \frac{\quad}{\quad} + \frac{\quad}{\quad}$$

$$\downarrow$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

2   3

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

(*) $5x-3 = A(x-3) + B(x+1)$

(i) $Ax + Bx = 5x$
(ii) $-3A + B = -3$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ can solve

Plug in values into (*) that cause one of the terms to be zero

Use +3 $\quad 5 \cdot 3 - 3 = B(\overset{3}{\cancel{x}}+1) \Rightarrow 12 = 4B \Rightarrow B = 3$

Use −1 $\quad -8 = -4A \Rightarrow A = 2$

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② Linear Repeated Factors

$$\int \frac{2x+5}{x^2-2x+1}\,dx = ?$$

$$\frac{2x+5}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

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$$2x+5 = A(x-1) + B$$

$$2x+5 = Ax + (B-A)$$

$$A=2 \qquad B-2=5 \Rightarrow B=7$$

$$\int \frac{2x+5}{x^2-2x+1}\,dx = \frac{2}{x-1} + \frac{7}{(x-1)^2}$$

$$= \frac{2(x-1)^{\cancel{2}\,1} + 7\cancel{(x-1)}}{(x-1)^{\cancel{3}\,2}}$$

$$= \frac{2x-2+\cancel{7x}+7}{(x-1)^2}$$

$$\frac{2x+5}{(x-1)^2}$$

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3 $$\int \frac{-2x+4}{(x^2+1)(x-1)^2}$$

$$(x^2+1)(x^2-2x+1) = \underline{x^4-2x^3+2x^2-2x+1}$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$(Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$$Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+$$

$$Cx^3-Cx^2+Cx-C+Dx^2+D = -2x+4$$

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x^3 $\quad A + C = 0 \Rightarrow \underline{A = -C}$ or $\boxed{-A = C}$

x^2 $\quad -2A + B - C + D = 0$

$-A + B + D = 0$

x $\quad A - 2B - A = -2 \Rightarrow -2B = -2$ or $B = 1$

1 $\quad B + A + D = 4$

$$\begin{array}{r} A + D = 3 \\ -A + D = -1 \\ \hline 2D = 2 \end{array} \Rightarrow D = 1$$

$1 + A + 1 = 4 \Rightarrow A = 2$

$C = -2$

$$\boxed{\begin{array}{l} A = 2 \\ B = 1 \\ C = -2 \\ D = 1 \end{array}}$$

(8)

$$\int -\frac{2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$\int \frac{2x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

$$\int \frac{du}{u} + \arctan x$$

$$\left[\ln|x^2+1| + \arctan x\right]$$

$u = x-1$

$du = dx$

$$2\int \frac{du}{u}$$

$$2\ln|x-1|$$

$u = x-1$

$du = dx$

$$\int \frac{du}{u^2}$$

$$-\frac{1}{x-1}$$

$-$

$+$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{dx}{\underline{x^4+2x^2+1}}$$

$$\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$1 = (Ax+B)(x^2+1) + Cx + D$$

$$\therefore 1 = Ax^3+Bx^2+Ax+B+Cx+D$$

1) $A+D = 1$

2) $A \equiv 0 \Rightarrow D = 1$

3) $B = 0$

4) $A+C = 0 \Rightarrow C = 0$

$A=0 \quad B=0 \quad C=0 \quad D=1$

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$$\int \frac{dx}{x(x^2+1)^2} = ?$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)x\underbrace{(x^2+1)}_{(x^3+x)} + (Dx+E)x$$

$$1 = A(x^4+2x^2+1) + (Bx^4+Cx^3+Bx^2+Cx) + Dx^2+Ex$$

x^4 $A+B=0$ $\boxed{(1)\ A=1}$

x^3 $C=0$ $\boxed{B=-1}$

x^2 $2A+B+D=0$ $\boxed{C=0}$ $\boxed{D=-1}$

x $C+E=0$ $\boxed{E=0}$

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} + \int \frac{-x}{x^2+1}\,dx + \int \frac{-x}{(x^2+1)^2}$$

$\ln|x|$

$u = x^2+1$
$du = 2x\,dx$

$$-\int \frac{1}{u}\left(\frac{du}{2}\right)$$

$$-\frac{1}{2}\int \frac{du}{u}$$

$$-\frac{1}{2}\ln|u|$$

$$-\frac{1}{2}\ln|x^2+1|$$

$u = x^2+1$
$du = 2x\,dx$

$$-\int \frac{1}{u^2}\frac{du}{2}$$

$$-\frac{1}{2}\int \frac{du}{u^2}$$

$$-\frac{1}{2}\left(-\frac{1}{u}\right) = \frac{1}{2u}$$

$$\frac{1}{2}\cdot\frac{1}{x^2+1}$$

$\ln|x|$ + $-\frac{1}{2}\ln|x^2+1|$ + $\frac{1}{2}\cdot\frac{1}{x^2+1}$

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Clear fractions:

$$x-1 = A(x+1)^2 + B(x+1) + C$$

Choose $x=-1$: $-1-1 = \boxed{C = -2}$

diff: $1 = A\left(2(x+1)\cdot 1\right) + B$

$1 = 2Ax + 2A + B \Rightarrow 2A + B = 1$

or let $x=-1$ in differentiated expression

to conclude $\boxed{B = 1}$

$1 = 2Ax + 2A + 1 \quad 2A(x+1) = 0$

$\Rightarrow \boxed{A = 0}$

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Ex 8.5

(1) $$\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x+2 = A(x-1) + B$$

Set $x=1$: $4 = A\cdot 0 + B \Rightarrow \boxed{B=4}$

$$2x+2 = A(x-1) + 4$$

$$2x = A(x-1) + 2$$

diff $\quad 2 = A \Rightarrow \boxed{A=2}$

$$\frac{2x+2}{(x-1)^2} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$$

Check: $$\frac{2\overbrace{(x^2-2x+1)}^{(x-1)^2} + 4(x-1)}{(x-1)^3}$$

$$= \frac{2(x-1)+4}{(x-1)^2} = \frac{2x+2}{(x-1)^2} \checkmark$$

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(22) $$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt = ?$$

$$\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$$

$$3t^2+t+4 = A(t^2+1) + Bt^2 + Ct$$

t^2 $\quad 3 = A + B$

t $\quad 1 = + C$

1 $\quad 4 = A$

$$\boxed{\begin{aligned} C &= 1 \\ A &= 4 \\ B &= -1 \end{aligned}}$$

$$= \int_1^{\sqrt{3}} \frac{4}{t}\,dt + \int_1^{\sqrt{3}} \frac{-t+1}{t^2+1}\,dt$$

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$$= 4\left(\ln t\right)\Big|_{1}^{\sqrt{3}} + \underline{\hspace{4cm}}$$

$$\int \frac{-t}{t^2+1}\,dt + \int \frac{1}{t^2+1}\,dt$$

$u = t^2+1$

$du = 2t\,dt$

$$\int \frac{1}{u}\left[-\left(\frac{du}{2}\right)\right]$$

$$-\frac{1}{2}\ln|u|$$

$$-\frac{1}{2}\ln|t^2+1|$$

$\arctan t$

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$$= 4\ln|t|\Big|_1^{\sqrt{3}} + \arctan t\Big|_1^{\sqrt{3}} - \frac{1}{2}\ln|t^2+1|\Big|_1^{\sqrt{3}}$$

$$\left(4(.54)-4\right)+\left(\frac{\pi}{3}-\frac{\pi}{4}\right)-\frac{1}{2}\left(1.38-0\right)$$

②

We can define a norm on a QNR as follows: given $a+b\sqrt{d}$, the norm $\left(N : \mathbb{Z}[\sqrt{d}] \to \mathbb{N}_{0+}\right)$ is defined as $|a^2-db^2|$.

N1: $N(x)=0$ iff $x=0$

N2: $N(xy)=N(x)N(y)$

N3: x is a unit iff $N(x)=1$

N4: $N(x)$ prime $\Rightarrow$ x irreducible

Verification of norm properties:

N1: $|a^2-db^2|=0 \Rightarrow a^2=db^2$

but this implies d not square free, so must be $a,b=0$

Other direction: $a+b\sqrt{d}=0$ then a,b are 0, hence $N(a+b\sqrt{d})=0$

Divisibility in Domains

We say if $a, b \in D$, that a & b are <u>associates</u> if $a = \underset{\cdot}{u} b$ $a\underset{\cdot}{u}^{-1} = b$

An <u>irreducible</u> in general is an element a, that is a non-zero non-unit such that if $a = bc$ then one of b or c is a unit.

A <u>prime</u> a in general is a non-zero non-unit such that if $a \mid bc$ then $a \mid b$ or $a \mid c$

Consider domains of form $\mathbb{Z}[\sqrt{d}]$, where $d \neq 1$ and d is square free

<u>QNR's</u> : quadratic number rings

N2: $N(xy) = N(x)N(y)$

Let $x = a + b\sqrt{d}$, $y = a' + b'\sqrt{d}$

$$N(xy) = N\left((a+b\sqrt{d})(a'+b'\sqrt{d})\right) =$$

$$N\left((aa' + bb'd) + \underline{(ab' + a'b)}\sqrt{d}\right)$$

$$= \left[(aa')^2 + \cancel{2aa'bb'd} + (bb')^2 d^2\right] -$$

$$\left[(ab')^2 + \cancel{2aa'bb'} + (a'b)^2\right] d$$

$$= \left[(aa')^2 + (bb')^2 d^2\right] - \left[(ab')^2 + (a'b)^2\right] d$$

Now calculate $N(x)N(y)$

$$(a^2 - db^2)(a'^2 - db'^2) =$$

$$(aa')^2 + d^2(bb')^2 - (a^2 b'^2 + a'^2 b^2) d$$

So... $N(xy) = N(x)N(y)$

N3: x is unit iff $N(x) = 1$

If $N(x) = 1$ $\left|a^2 - db^2\right| = 1$

$|a - \sqrt{d}\, b| \cdot \underline{|a + \sqrt{d}\, b|}$ but then

$a + b\sqrt{d}$ has inverse, i.e. is unit.

$\left[N(x)\right]^2 = 1 \Rightarrow N(x) = 1$

N4: $N(x)$ prime $\Rightarrow$ x irreducible

$|a^2 - b^2 d|$ is prime

If $a + b\sqrt{d}$ factors non-trivially

$(a + b\sqrt{d}) = (x + y\sqrt{d})(z + w\sqrt{d})$

$N(\uparrow) = N(\uparrow) \cdot N(\uparrow)$

$p = \underline{p} \cdot \underline{1}$

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Consider $\mathbb{Z}[\sqrt{-3}]$.

Let $a+b\sqrt{-3}$ have norm $|a^2+3b^2|$

Suppose this factors as xy

where neither x nor y is a unit.

Recall $N(xy) = N(x)N(y)$

Take $1+\sqrt{-3} \in \mathbb{Z}[\sqrt{-3}]$

$a=b=1$

$N(1+\sqrt{-3}) = |1^2+3(+1)| = |4|$

$N(x)N(y) = N(xy) = 4 = 2 \cdot 2$

But if $N(x) = 2$ $\quad a^2+3b^2=2$

Only other possibility is $N(xy) = 4 \cdot 1$

Then this forces one of $N(x)$ or $N(y)$

to be 1, hence x or y is a unit...

then $1+\sqrt{-3}$ is irreducible over

$\mathbb{Z}[\sqrt{-3}]$.

⑥

But $1+\sqrt{-3}$ is not a prime.

If prime it divides 2

$$\underline{(1+\sqrt{-3})}(1-\sqrt{-3}) = 4 = \underline{2 \cdot 2}$$

Consider $\underbrace{(1+\sqrt{-3})(a+b\sqrt{-3})} = 2$

$$(a-3b) + \sqrt{-3}\,(a+b) = 2$$

$$a+b \to 0$$

Must find a, b ∋ $a-3b=2$

$$-4b = 2$$

$$b = -\tfrac{1}{2} \Rightarrow\!\!\!\!\!\times$$

Given domain D, if $a \in D$ and a is prime, a is irreducible

Pf: Suppose $a = bc$ Need to show either b or c is a unit

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a being prime means $a|b$ or $a|c$

so let a divide b, then

$b = at$.

$$a = \underbrace{at}_{b}c = a = a(tc)$$

$\Rightarrow tc = 1$ so c is a unit

but that means the factorization of a is essentially trivial, so a is an irreducible.