3-18

Trig Integrals

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \sin^m x \cdot \cos^n x\, dx \qquad m, n \in \mathbb{N}$$

Ex: $$\int \underbrace{\sin^m x}_{u^m} \underbrace{\cos x\, dx}_{du = d(\sin x)} = \frac{\sin^{m+1} x}{m+1} + C$$

If m is even or odd, n is odd, mimic the above

$$\int \sin^m x \cos^3 x\, dx \rightarrow \int \sin^m x \underbrace{\cos^2 x}_{1-\sin^2 x} \underline{(\cos x\, dx)}$$

save 1 power for differential

$1-\sin^2 x$ by Pythagorean

$$= \int \sin^m x \left(1-\sin^2 x\right) d(\sin x)$$

$$\int \sin^m x \, d(\sin x) - \int \sin^{m+2} x \, d(\sin x)$$

$u^m du \qquad u^{m+2} du$

$$= \frac{\sin^{m+1} x}{m+1} - \frac{\sin^{m+3} x}{m+3} + C$$

m is odd ..

Ex: n is even - check m

$$\int \sin^{\overset{\text{odd}}{m}} x \cos^{\overset{\text{even}}{n}} x \, dx =$$

$$\int \sin^{m-1} x \cos^n x \overbrace{(\sin x \, dx)}^{-d(\cos x)}$$

Now m-1 is even - use Pythagorean Thm to convert rest of sines into cosines

Ex: $\int \sin^3 x \cos^8 x \, dx =$

$$\int \underset{\uparrow}{\sin^2 x} \cos^8 x \underbrace{(\sin x \, dx)}_{-d(\cos x)}$$

$1 - \cos^2 x$

$$\int \sin^3 x \cos^8 x \, dx = \int (1-\cos^2 x)\cos^8 x \,(-d(\cos x))$$

$$= -\int \cos^8 x \, d(\cos x) + \int \cos^{10} x \, d(\cos x)$$

$u = \cos x$ $\quad -\int u^8 \, du + \int u^{10} \, du$

$$-\frac{\cos^9 x}{9} + \frac{\cos^{11} x}{11} + C$$

What about both m, n being even.

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

half-angle formulas

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos 2\theta}{2}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos 2\theta}{2}}$$

Ex. $\int \sin^2 x \cos^4 x \, dx$

$\downarrow$

$$\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\,dx$$

$$= \frac{1}{8}\int (\overset{(i)}{1} + \overset{(ii)}{\cos 2x} - \overset{(iii)}{\cos^2 x} - \overset{(iv)}{\cos^3 x})\,dx$$

(i) x

(ii) $\frac{\sin 2x}{2}$

(iii) $-\int \cos^2 x \, dx = -\int \frac{1+\cos 2x}{2} dx$

$$\boxed{-\frac{x}{2} - \frac{\sin 2x}{4}}$$

(iv) $-\int \cos^3 x\,dx = -\int (1-\sin^2 x)\,d(\sin x)$

$$-\sin x + \frac{\sin^3 x}{3}$$

$$\int \sin^2 x \cos^4 x\,dx = \frac{1}{8}\left[\cancel{x} + \cancel{\frac{\sin 2x}{2}} \pm \frac{x}{2} \mp \frac{\sin 2x}{4} - \sin x + \frac{\sin^3 x}{3}\right]$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2}\cos 2x\,dx$$

$$\sqrt{1+\cos 2(2x)} = \sqrt{2\cos^2(2x)} = \sqrt{2}\cos 2x$$

$$\sqrt{2}\int_0^{\pi/4} \cos 2x\,dx = \frac{\sqrt{2}}{\cancel{2}}\left[\frac{\sin 2x}{\cancel{2}}\right]_0^{\pi/4} = \sqrt{2}$$

$$\int \sec^3 x\,dx = ? = \int \underbrace{\sec x}_{u}\,(\underbrace{\sec^2 x\,dx}_{dv})$$

$u = \sec x$

$v = \tan x$

$du = \sec x \tan x\,dx$

$dv = \sec^2 x\,dx$

$$\cancel{\sec x \tan x - \int \tan x \tan^2 x\,dx}$$

(uv)

By Parts: $\sec x \tan x - \int \tan x\,(\sec x \tan x)\,dx$

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{(\sec^2 x - 1)}\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \underbrace{\int \sec^3 x\,dx}_{\text{LHS}} + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$\downarrow$$

$$\ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

$$\sec^2 x = 1 + \tan^2 x$$

$$d(\tan x) = \sec^2 x\,dx$$

$$\int \tan^4 x \sec^4 x\,dx$$

$$= \int \tan^4 x \sec^2 x \underbrace{(\sec^2 x\,dx)}_{d(\tan x)}$$

$$= \int \tan^4 x (1 + \tan^2 x)\,d(\tan x)$$

$$= \int \tan^4 x\,d(\tan x) + \int \tan^6 x\,d(\tan x)$$

$$\boxed{\frac{\tan^5}{5} + \frac{\tan^7}{7} + C}$$

Ex 8.3

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4 $\displaystyle\int \sin^4 2x \underline{\cos 2x\, dx} = \boxed{\frac{\sin^5 2x}{10} + C}$

$d(\sin 2x) = 2\cos 2x\, dx$

$\displaystyle\int \sin^4 2x \frac{d(\sin 2x)}{2} = \frac{1}{2}\int u^4 du = \frac{u^5}{10} + C$

Let $u = \sin 2x$

10 $\displaystyle\int_0^{\pi/6} 3\cos^5 3x\, dx =$

$\frac{d(\sin 3x)}{3}$

$\displaystyle 3\int (\cos^4 3x)(\cos 3x\, dx)$

$(1 - \sin^2 3x)^2 = 1 - 2\sin^2 3x + \sin^4 3x$

$\displaystyle = \cancel{3}\int \left[1 - 2\sin^2 3x + \sin^4 3x\right] d\left(\frac{\sin 3x}{\cancel{3}}\right)$

$\displaystyle = \int d(\sin 3x) - 2\int \sin^2 3x\, d(\sin 3x) + \rightarrow$

⑨

$\int \sin^4 3x \, d(\sin 3x)$

$$= \boxed{\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5} + C}$$

⑯ $\int 7\cos^7 t \, dt = 7\int \cos^6 t \, \underbrace{(\cos t \, dt)}_{d(\sin t)}$

$$= 7\int (1-\sin^2 t)^3 \, d(\sin t)$$

$$= 7\int \left[1 - 3\sin^2 t + 3\sin^4 t - \sin^6 t\right] d(\sin t)$$

$$\boxed{(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3}$$

$$7\left[\sin t - \frac{\cancel{3}\sin^3 t}{\cancel{3}} + \frac{3\sin^5 t}{5} - \frac{\sin^7 t}{7}\right]$$

$$\boxed{7\sin t - 7\sin^3 t + \frac{21}{5}\sin^5 t - \sin^7 t}$$ Ⓐ

25 $$\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt = \int_0^{\pi} \cos t\,dt = \Big[\sin t\Big]_0^{\pi} = 0$$

33 $$\int \sec^2 x \tan x\,dx = \int \tan x\, d(\tan x)$$

$$= \frac{\tan^2 x}{2} + C$$

36 $$\int \sec^3 x \tan^3 x\,dx = \int \sec^2 x \overbrace{\tan^2 x}^{\sec^2 x - 1} \underbrace{(\sec x \tan x\,dx)}_{d(\sec x)}$$

$$= \int \sec^2 x\,(\sec^2 x - 1)\,d(\sec x)$$

$$= \int (\sec^4 x - \sec^2 x)\,d(\sec x)$$

$$\boxed{= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

40 $\int e^x \sec^3(e^x)\,dx$

$= e^x\sec^3(e^x) - \int e^x\, 3\cdot\sec^2(e^x)\cdot e^x\,dx$

$e^x\sec^3(e^x) - 3\int e^{2x}\sec^2(e^x)\,dx$

$\frac{d(\tan(e^x))}{dx} = \sec^2(e^x)\cdot e^x$

$= e^x\sec^3(e^x) - 3\int e^x\, d(\tan(e^x))$

d/dx	$\int dx$
e^x	$d(\tan e^x)$
e^x	$\tan e^x$
e^x –	$e^x\sec^2 e^x$

62 $\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 3\theta = \sin(\theta + 2\theta) = ?$

$\sin\theta\cos 2\theta + \cos\theta\sin 2\theta$

$\sin\theta \cdot \sin 2\theta = \boxed{2\sin^2\theta\cos\theta.}$

~~$2\sin^3\theta\cos^2\theta +$~~

$\sin\theta(\cos^2\theta - \sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$

$\sin\theta\cos^2\theta - \sin^3\theta + 2\sin\theta\cos^2\theta$

$\sin\theta(1-\sin^2\theta) - \sin^3\theta + 2\sin\theta(1-\sin^2\theta)$

$\sin\theta - \sin^3\theta - \sin^3\theta + 2\sin\theta - 2\sin^3\theta$

$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

$$\sin\theta \sin 2\theta \sin 3\theta = \left[2\sin^2\theta\cos\theta\right]\cdot\left[3\sin\theta - 4\sin^3\theta\right]$$

$$= 6\sin^3\theta\,\underline{\cos\theta} - 8\sin^5\theta\,\underline{\cos\theta}$$

$$\int \quad = 6\int \sin^3\theta\, d(\sin\theta) - 8\int \sin^5\theta\, d(\sin\theta)$$

$$= 6\,\frac{\sin^4\theta}{4} - \frac{8\sin^6\theta}{6}$$

$$\boxed{= \frac{3}{2}\sin^4\theta - \frac{4}{3}\sin^6\theta + C}$$