Integration by Substitution

$$\int f(x)\,dx \xrightarrow{x(t)} \int f(x(t))\,\frac{dx}{dt}\,dt$$

hard doable

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(1) $\displaystyle\int 2(2x+4)^5\,dx = ?$

Let $u = 2x+4 \Rightarrow \dfrac{du}{dx} = 2$, $du = 2dx$

$$\int u^5\,du = \frac{u^6}{6} + C \rightsquigarrow \boxed{\frac{(2x+4)^6}{6}}$$

(3) $\displaystyle\int 2x(x^2+5)^{-4}\,dx = ?$

Let $u = x^2+5$ $\quad du = 2x\,dx$

$$\int u^{-4}\,du = \frac{u^{-3}}{(-3)} \rightsquigarrow \boxed{-\frac{1}{3(x^2+5)^3}}$$

5. $\int (3x+2)(3x^2+4x)^4\,dx = ?$

Let $u = 3x^2+4x$ $\quad du = (6x+4)\,dx$

$$\int u^4 \cdot \frac{du}{2} = \frac{1}{2}\int u^4\,du = \frac{1}{2}\,\frac{u^5}{5}$$

$$\frac{1}{10}\left(3x^2+4x\right)^5$$

6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx = ?$

Let $u = (1+\sqrt{x})$ $\quad du = \frac{dx}{2\sqrt{x}}$

$$\int u^{1/3}(2\,du) = 2\int u^{1/3}\,du = 2\cdot\frac{3}{4}\cdot u^{4/3}$$

$$\frac{3}{2}u^{4/3} = \boxed{\frac{3}{2}\left(1+\sqrt{x}\right)^{4/3}}$$

7 $\int \sin 3x\,dx = ?$

Let $u = 3x \qquad du = 3dx$

$$\int \sin u \frac{du}{3} = \frac{1}{3}\int \sin u\,du$$

$$\frac{1}{3}(-\cos u) = -\frac{\cos 3x}{3}$$

8 $\int x \sin(2x^2)\,dx = ?$

Let $u = 2x^2 \qquad du = 4x\,dx$

$$\frac{1}{4}\int \sin u\,du = \frac{1}{4}(-\cos u) = -\frac{1}{4}\cos(2x^2)$$

(10) $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt$

Let $u = 1-\cos\frac{t}{2}$ $\quad du = \sin\frac{t}{2}\left(\frac{1}{2}\right)$

$du = \frac{1}{2}\sin\frac{t}{2}$

$= 2\int u^2\,du = \frac{2}{3}u^3 \rightsquigarrow \frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3$

(11) $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}$

Let $u = 1-r^3$ $\quad du = -3r^2\,dr$

$\frac{-3}{-3}\int (1-r^3)^{-1/2}\, r^2\,dr = -\frac{1}{3}\int u^{-1/2}\,du$?

$-\frac{1}{3}\left(2u^{1/2}\right) \rightsquigarrow -\frac{2}{3}(1-r^3)^{1/2}$

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(13) $\int \sqrt{x}\,\sin^2\left(x^{3/2}-1\right)dx$

Let $u = x^{3/2}-1$ $\qquad du = \frac{3}{2}\sqrt{x}\,dx$

$\frac{2}{3}\int \sin^2 u \cdot du = ?$

Short Trig Lesson

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

Recall $\cos^2\theta + \sin^2\theta = 1$

$\cos 2\theta = 1-\sin^2\theta - \sin^2\theta = \underline{1-2\sin^2\theta}$

$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta) = \underline{2\cos^2\theta - 1}$

Start w/ $\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta - 1 = -2\sin^2\theta$

$$\boxed{\frac{1-\cos 2\theta}{2} = \sin^2\theta}$$

$$\frac{2}{3}\int \underbrace{\frac{1-\cos 2u}{2}}_{\text{half-angle formula}}\,du = \frac{2}{3}\left[\int \frac{du}{2} - \int \frac{\cos 2u\,du}{2}\right]$$

(15) $$\int \csc^2 2\theta \cot 2\theta \, d\theta = \int \csc 2\theta \left(\underline{\csc 2\theta \cot 2\theta}\right) d\theta$$

Recall $(\csc 2\theta)' = 2\csc 2\theta \cot 2\theta$

Let $u = \csc 2\theta \rightarrow$

$du = 2\csc 2\theta \cot 2\theta \, d\theta$

$$\int u \frac{du}{2} = \frac{u^2}{4} \rightsquigarrow \frac{\csc^2 2\theta}{4}$$

$$\int \frac{1}{\sin^2 2\theta} \cdot \frac{\cos 2\theta}{\sin 2\theta}\, d\theta = \int \frac{\cos 2\theta \, d\theta}{\sin^3 2\theta}$$

⑦ Recall: $\left[e^{g(x)}\right]' = e^{g(x)} \cdot g'(x)$

(51) $\int \cos x \, e^{\sin x} dx$

$u = e^{\sin x} \qquad du = e^{\sin x} \cdot \cos x \, dx$

$\int du = u \longrightarrow e^{\sin x}$

(54) $\int \underline{\frac{1}{x^2} e^{1/x}} \sec(1+e^{1/x}) \tan(1+e^{1/x}) \underline{dx}$

Recall $(\sec u)' = \sec u \tan u$

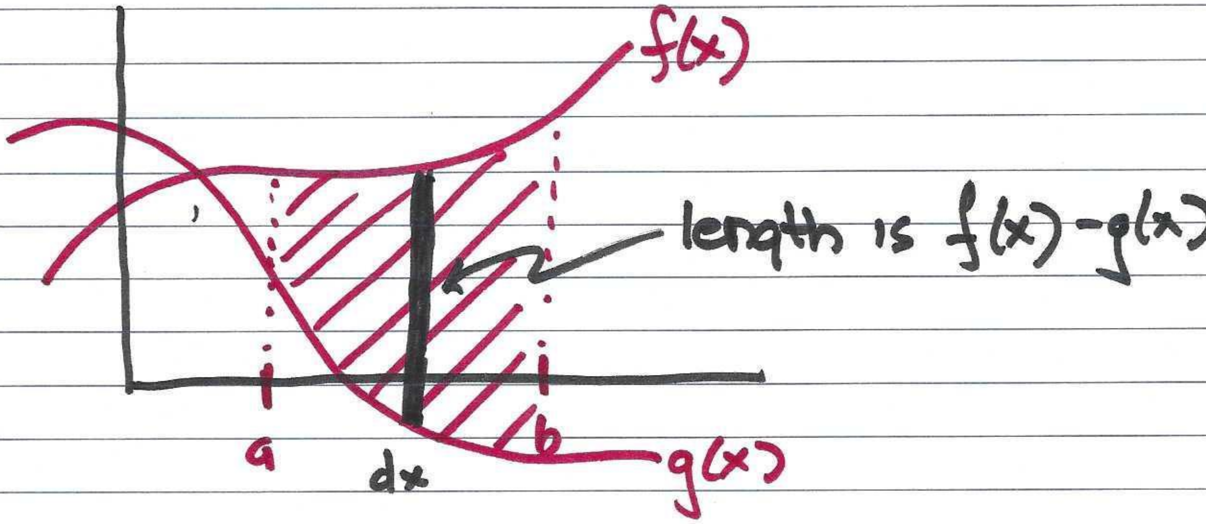
Let $u = 1 + e^{1/x} \qquad \frac{du}{dx} = e^{1/x} \cdot \left(-\frac{1}{2x^2}\right)$

$du = -\frac{1}{2}\left(e^{1/x} \cdot \frac{1}{x^2}\right) dx$

$-2\int (\sec u)' du = -2 \ln(\sec u + \tan u)$

★★ $= -2 \ln\left[\sec(1+e^{1/x}) + \tan(1+e^{1/x})\right]$

Area between curves



$$dA = \left[f(x) - g(x)\right]dx$$

$$\text{Area} = \int_a^b \left(f(x) - g(x)\right) dx \quad \Bigg| = \int_a^b \left|f(x) - g(x)\right| dx$$

Let $f(x) = 2 - x^2$; $g(x) = -x$

What is area between f & g from $x = -1$ to $+2$

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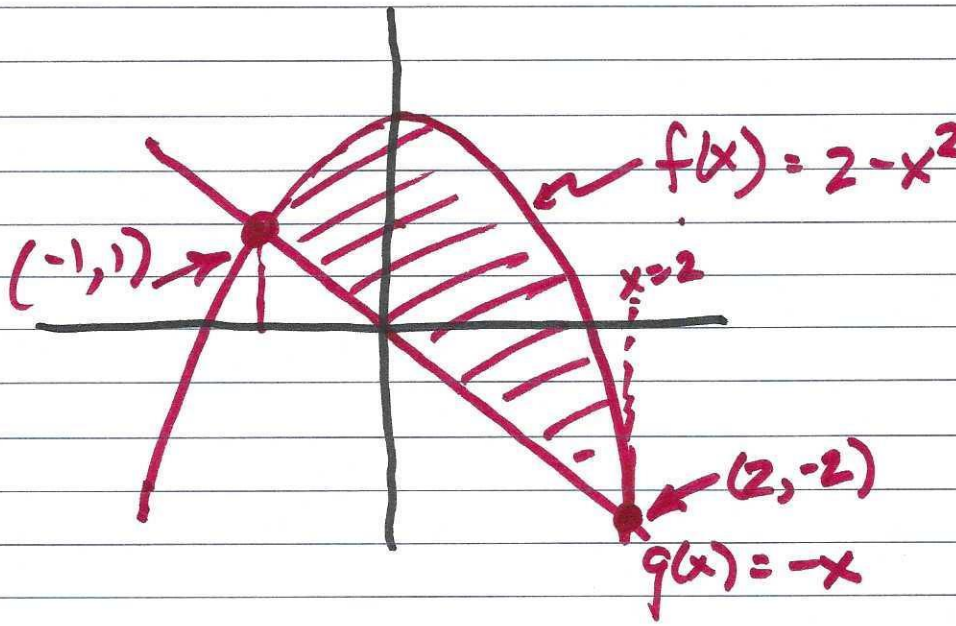

$2-x^2=-x \qquad x^2-x-2 \rightsquigarrow (x-2)(x+1)$

$x=2 \qquad x=-1$

$$\text{Area} = \int_{-1}^{2} \underbrace{(2-x^2)}_{f} - \underbrace{(-x)}_{g}\,dx =$$

$$\int_{-1}^{2} (-x^2+x+2)\,dx =$$

$$\left[-\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{2}$$

$$\left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)=$$

$$3\frac{1}{3}-1\frac{1}{6}=3\frac{2}{6}-1\frac{1}{6}=\cancel{2\frac{1}{6}}$$

Check this ??

$$\int_0^3 \sqrt{y+1}\,dy \rightsquigarrow \int_{y=0}^{3} u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_{y=0}^{y=3}$$

Let $u=y+1$ $\quad du=dy$

$$\left[\frac{2}{3}(y+1)^{3/2}\right]_0^3 = \frac{2}{3}\left[(y+1)^{3/2}\right]_0^3 =$$

$$\frac{2}{3}(8-1)=\frac{14}{3}$$

$$\int_0^{\pi/4} \tan x \sec^2 x \, dx$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$

$$\int_{x=0}^{x=\pi/4} u \, du = \left[\frac{u^2}{2}\right]_{x=0}^{x=\pi/4} = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\int_0^{\pi} 3 \cos^2 x \underline{\sin x \, dx}$$

Let's try $u = \cos x \quad du = -\sin x \, dx$

$$3\int_{x=0}^{\pi} u^2(-du) = -3\int_{x=0}^{\pi} u^2 du = -3\left[\frac{u^3}{3}\right]_1^{-1}$$

$$-3\left(-\frac{1}{3} - \frac{1}{3}\right) = 2$$

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$$\int_0^{\pi^{2/3}} \sqrt{\theta}\cos^2\left(\theta^{3/2}\right)d\theta$$

Let $u = \theta^{3/2}$ $du = \frac{3}{2}\sqrt{\theta}\, d\theta$

$$\int_0^{\pi^{2/3}} \cos^2 u \left(\frac{2}{3}du\right) = \frac{2}{3}\int_{\theta=0}^{\theta=\pi^{2/3}} \cos^2 u\, du$$

$\sqrt{\theta}\,d\theta$

$\cos^2 u = \frac{1+\cos 2u}{2}$ (half-angle formula)

$$= \frac{2}{3}\int_{u=0}^{u=\pi}\left(\frac{1+\cos 2u}{2}\right)du = \frac{2}{3}\left[\int_0^{\pi}\frac{du}{2} + \int_0^{\pi}\frac{\cos 2u}{2}du\right]$$

= homework