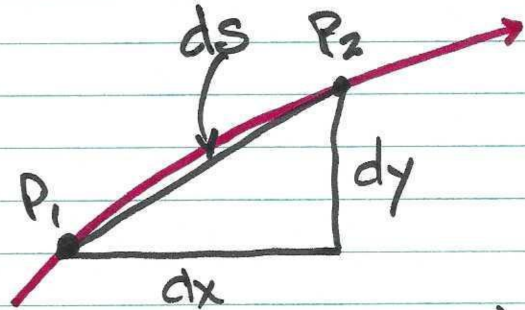
1-21

Rectification of Curves
(finding length of curves)



$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + \frac{dy^2}{dx^2}} \cdot dx$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{or} \quad ds = \sqrt{1 + [f'(x)]^2}\, dx$$

Example

Curve is $y(x) = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \quad (0 \le x \le 1)$

Then $\frac{dy}{dx} = \frac{\cancel{4}^{\,2}\sqrt{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} x^{1/2} = 2\sqrt{2}\, x^{1/2} = 2\sqrt{2x}$

$$\left(\frac{dy}{dx}\right)^2 = 8x \Rightarrow \sqrt{1 + 8x}\, dx = ds$$

Length is $\int_0^1 \sqrt{1+8x}\,dx$

Let $u = 1+8x \Rightarrow du = 8\,dx$

$$\text{Length} = \int_{u=1}^{u=9} u^{1/2}\frac{du}{8} = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{8}\frac{2}{3}(27-1)$$

$$= \frac{52}{3\cdot 8} = \frac{52}{24} = \boxed{\frac{13}{6}}$$

$y = \cosh x$ find length between $x=0$ & $x=2$

$y' = \sinh x$

Note: $\cosh^2 x = 1+\sinh^2 x$

$\sqrt{1+\sinh^2 x} = ?$

$\downarrow$

$\cosh x$

$$\text{Length} = \int_0^2 \cosh x\,dx = \left[\sinh x\right]_0^2 = ?$$

$$\left[\frac{e^x - e^{-x}}{2}\right]_0^2 = 0 + \frac{e^2 - e^{-2}}{2} \approx 3.6$$

2 $y = \sqrt{x^3} \qquad x = 0 \text{ to } x = 4$

$= x^{3/2}$

So... $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$

?

Then $ds = \sqrt{1 + \frac{9}{4}x} \, dx$

$$\text{Length} = \int_0^4 \sqrt{1 + \frac{9}{4}x} \, dx$$

Let $u = 1 + \frac{9}{4}x \Rightarrow du = \frac{9}{4}dx$

Rewrite $\int$: $\text{Length} = \int_1^{10} \sqrt{u} \, \frac{4}{9} \, du$

$$\text{Length} = \frac{4}{9}\int_1^{10} u^{1/2} du = \frac{4}{9}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{10} =$$

$$\boxed{\frac{8}{27}\left(10^{3/2} - 1\right)}$$

12 $y = \frac{x^3}{3} + \frac{1}{4x}$ $\qquad 1 \le x \; 3$

$$\frac{dy}{dx} = x^2 + \frac{1}{4}\left(-x^{-2}\right) = \left(x^2 - \frac{1}{4x^2}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^4 - 2\left(\frac{1}{4}\right) + \frac{1}{16x^4}$$

$$= x^4 + \frac{1}{16x^4} + \frac{1}{2}$$

$$\frac{\sqrt{16x^8 + 8x^4 + 1}}{4x^2} \qquad \frac{16x^8 + 8x^2 + 1}{16x^4}$$

$$16x^8 + 8x^4 + 1 = \left(4x^4 + 1\right)^2$$

So... $\displaystyle \boxed{\int_1^3 \frac{\left(4x^4 + 1\right)^2}{4x^2}\,dx}$

(27) $y = \int_0^x \sqrt{\cos 2t}\, dt \qquad 0 \le x \le \frac{\pi}{4}$

FTC: $\frac{dy}{dx} = \sqrt{\cos 2x}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x$

$\cos^2 \frac{\theta}{2} = \frac{1+\cos 2\theta}{2}$

$\cos \frac{\theta}{2} = \sqrt{\frac{1+\cos 2\theta}{2}}$

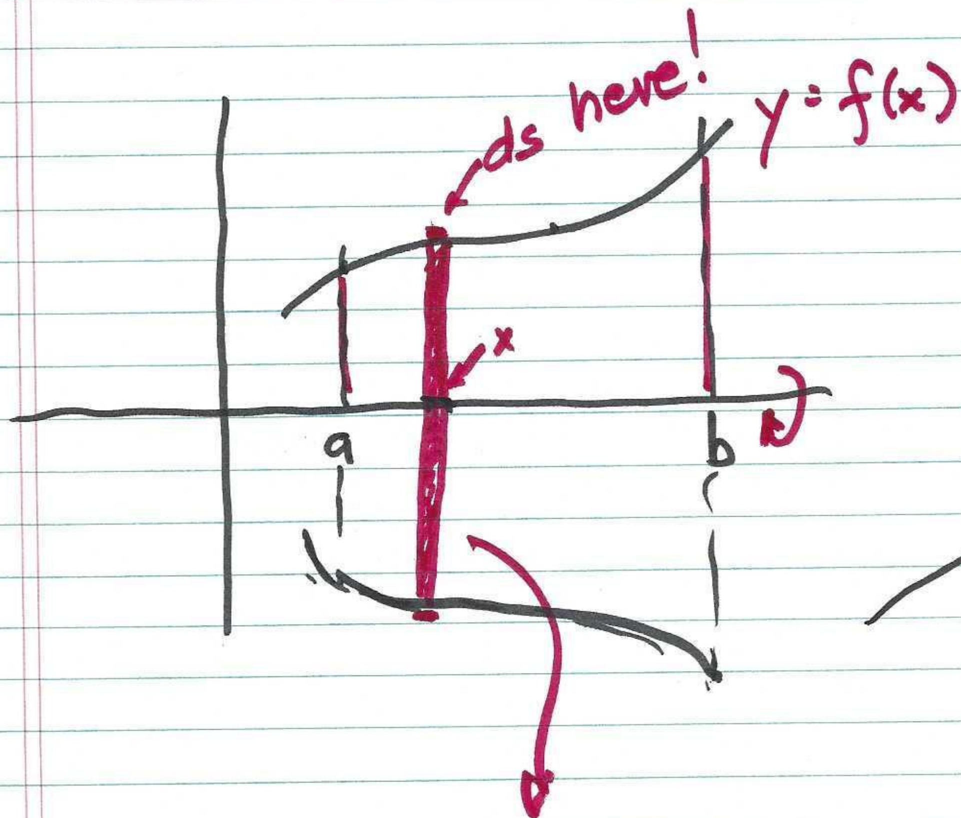
$\sqrt{2} \cos \frac{\theta}{2} = \sqrt{1+\cos 2\theta}$

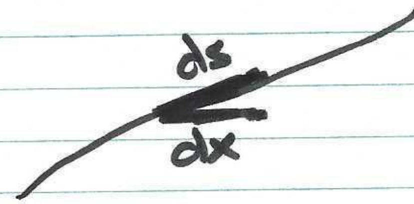
$L = \int_0^{\pi/4} \sqrt{1+\cos 2x}\, dx$

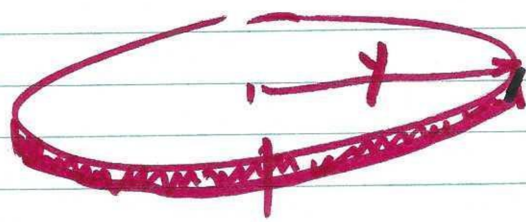
$L = 2\int_0^{\pi/4} \sqrt{2} \cos\left(\frac{x}{2}\right) d\left(\frac{x}{2}\right)$

$= \sqrt{2} \cdot 2 \int_0^{\pi/4} \underline{\cos\left(\frac{x}{2}\right) d\left(\frac{x}{2}\right)} = 2\sqrt{2}\left[\sin \frac{x}{2}\right]_0^{\frac{\pi}{4}}$

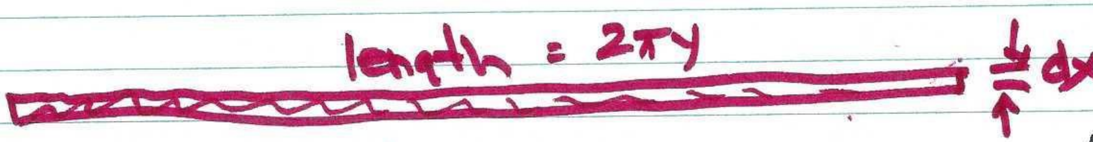
$= 2\sqrt{2} \sin\left(\frac{\pi}{8}\right)$





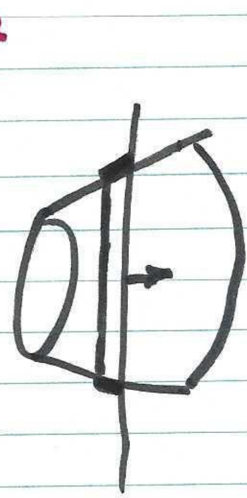


$ds = \sqrt{1+(y')^2}\,dx$



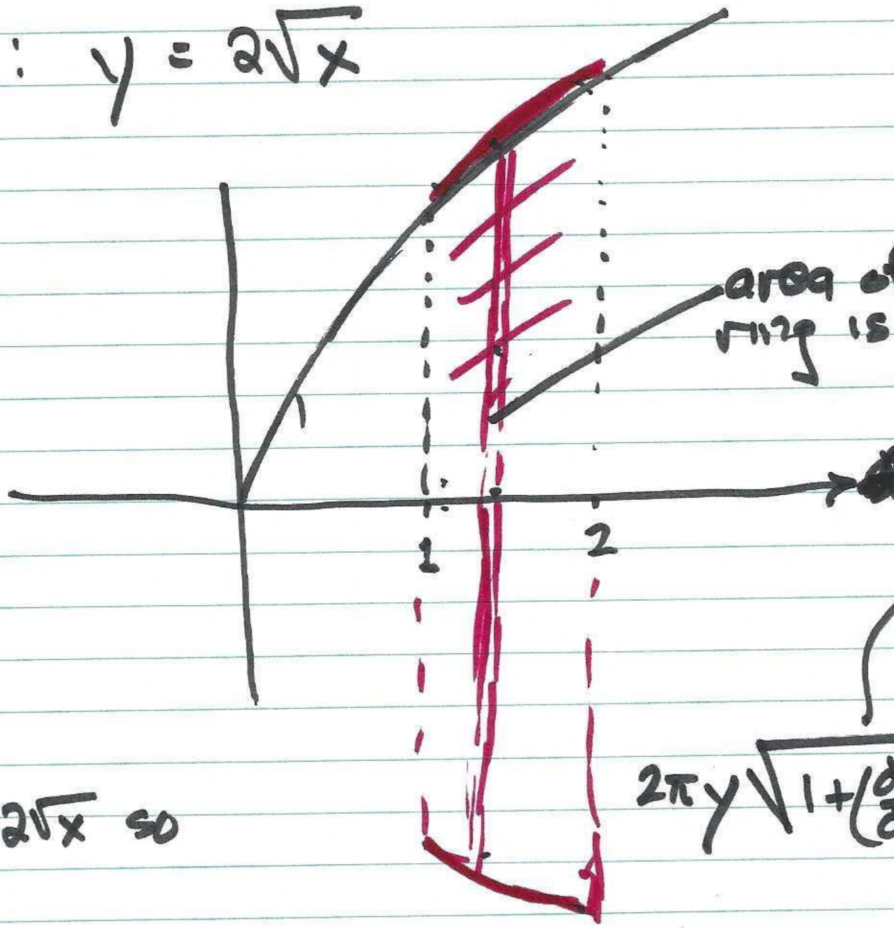
want "slant area"

$$\text{Surface Area} = \int_a^b 2\pi f(x)\,dx$$



$$\text{Surface Area} = 2\pi \int_a^b y\, ds = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Example: $y = 2\sqrt{x}$



area of differential ring is

$$2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = dA$$

$y = 2\sqrt{x}$ so

$\frac{dy}{dx}$

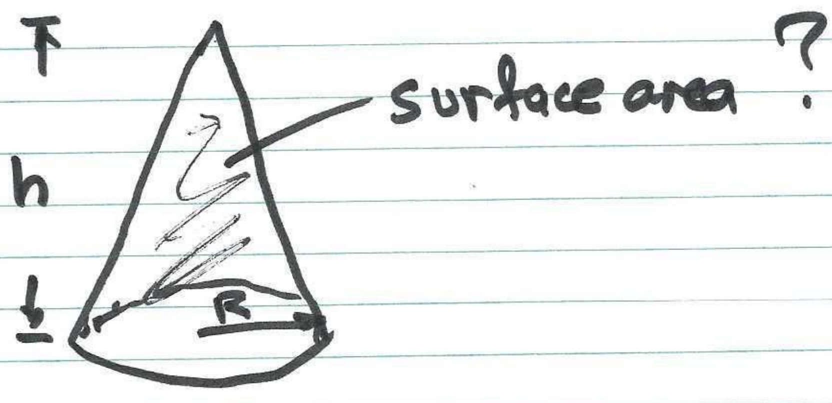
$$SA = \int_1^2 2\pi \left(2\sqrt{x}\right) \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\, dx$$

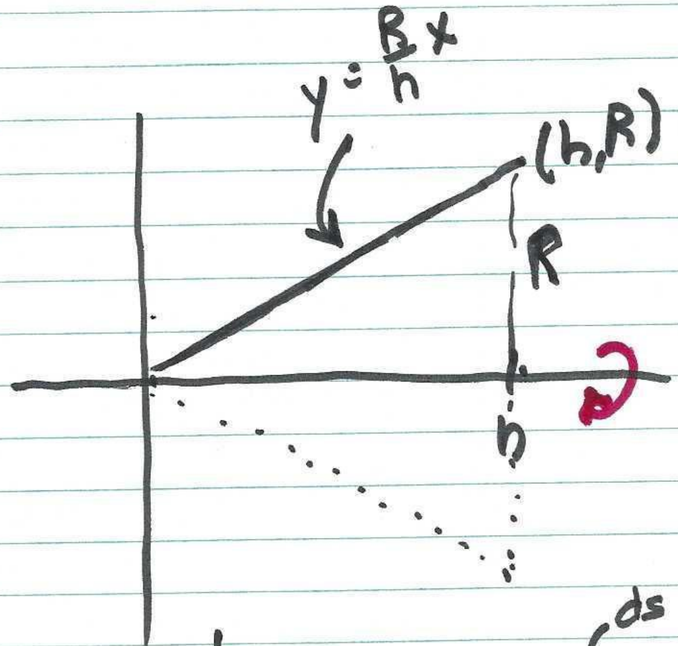
$$\hookrightarrow \sqrt{1+\frac{1}{x}} = \sqrt{\frac{x+1}{x}}$$

$$SA = 4\pi \int_1^2 \cancel{\sqrt{x}} \sqrt{\frac{x+1}{\cancel{x}}}\, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1}\, d(x+1) = (4\pi)\left(\frac{2}{3}\right)(x+1)^{3/2}$$

$$= \frac{8\pi}{3}\Big[(x+1)^{3/2}\Big]_1^2 = \underline{\frac{8\pi}{3}\left(3^{3/2} - 2^{3/2}\right)}$$





$$\frac{dy}{dx} = \left(\frac{R}{h}\right)$$

$$\sqrt{\frac{R^2+h^2}{h^2}}$$

$$SA = \int_0^h 2\pi\left(\frac{R}{h}x\right)\overbrace{\left(\sqrt{1+\left(\frac{R}{h}\right)^2}\right)}^{ds}dx$$

$$= 2\pi\frac{R}{h}\cdot\left(\sqrt{\frac{R^2+h^2}{h^2}}\right)\int_0^h x\,dx$$

$$= \cancel{2}\pi\frac{R}{\cancel{h^2}}\sqrt{R^2+h^2}\left[\frac{\cancel{x^2}}{\cancel{2}}\right]_0^h =$$

$$= \pi R\sqrt{R^2+h^2} \text{ is } \underline{SA}$$

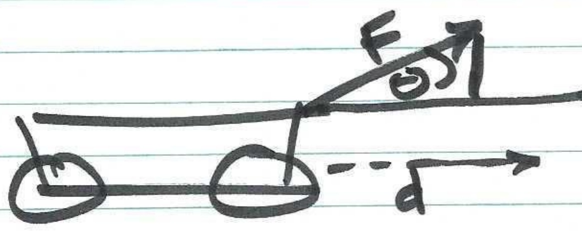
Preview of 6.5 - Work

If force F acts thru distance d, then work performed by force is:

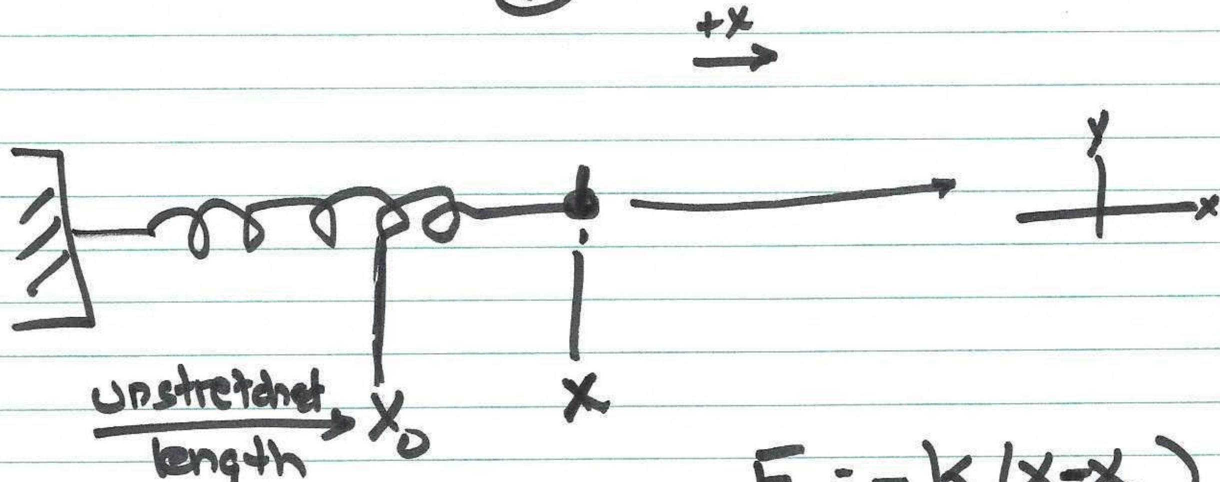
$$W = Fd$$

Differentially $dW = F \cdot dx$

Force must be aligned with displacement or some factor must be used to express the amount of force aligned with displacement.



Work done is $F \cdot d \cdot \cos\theta$



$$F = -k(x - x_0)$$

spring constant

"stretch"

Spring Law : $F(x) = -k(x - x_0)$

$$F(x)\,dx = dW(x) \quad \boxed{\frac{dW(x)}{dx} = F(x)}$$

$$\text{Total work} = \int_{\text{start}}^{\text{stop}} -k(x - x_0)\,dx$$

$$W = \frac{1}{2}k(x - x_0)^2$$