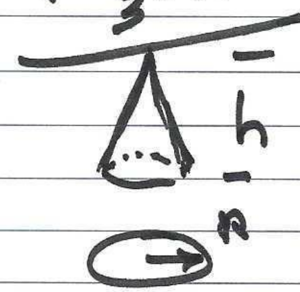
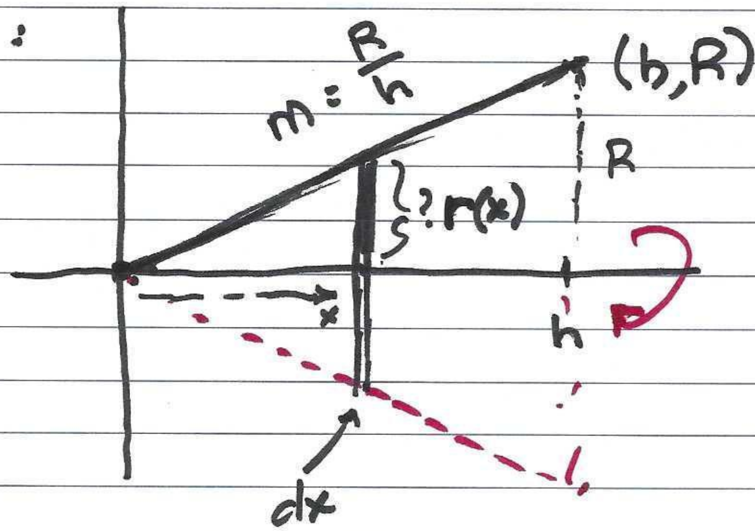
Volumes by Cross-Sections

$V = \frac{1}{3}\pi R^2 h$



Ex 1:



Disk Method

$$\frac{r(x)}{x} = \frac{R}{h} \Rightarrow r(x) = \frac{Rx}{h}$$

What is area @ x $\quad A(x) = \pi\left(\frac{Rx}{h}\right)^2$

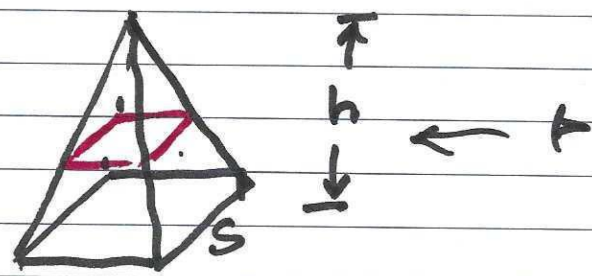
$$dV(x) = A(x)\,dx = \frac{\pi R^2 x^2}{h^2}\,dx$$

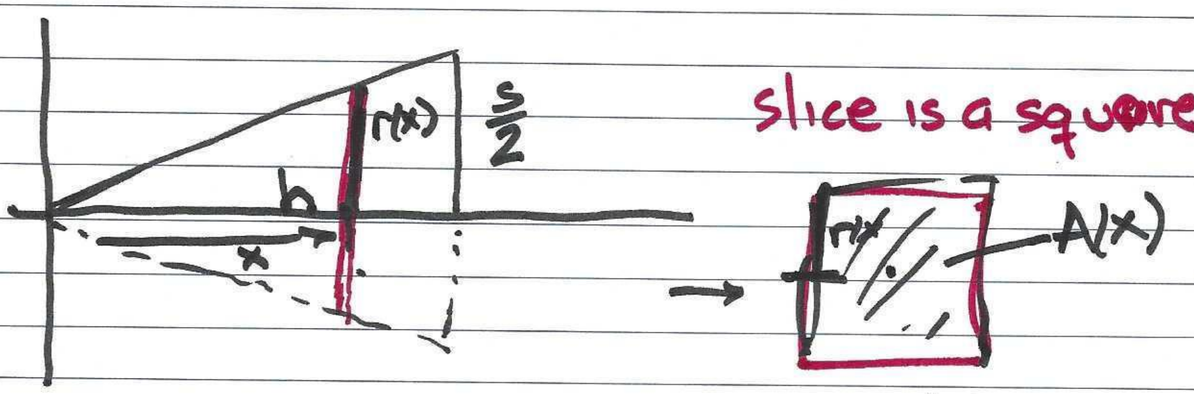
$$Vol = \int_0^h \frac{\pi R^2}{h^2} x^2\,dx = \frac{\pi R^2}{h^2}\int_0^h x^2\,dx$$

$$Vol = \frac{\pi R^2}{h^2} \cdot \int_0^h x^2 dx = \frac{\pi R^2}{h^2}\left[\frac{x^3}{3}\right]_0^h =$$

$$\frac{\pi R^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi R^2 h$$

Ex 2 :





$$\frac{r(x)}{x} = \frac{s/2}{h} \Rightarrow r(x) = \frac{s/2}{h} \cdot x$$

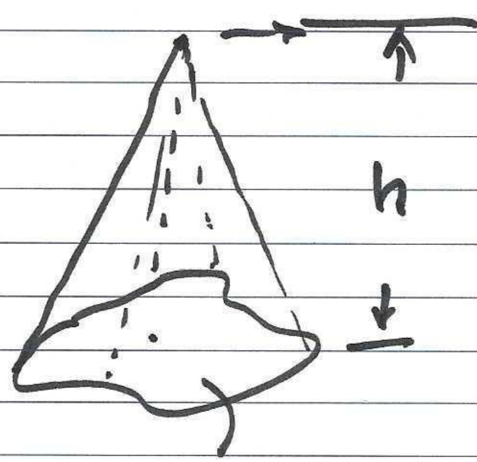
$$A(x) = \left[2r(x)\right]^2 = \frac{s^2}{h^2}x^2$$

$$dV(x) = A(x)\,dx = \frac{s^2}{h^2}x^2\,dx$$

$$\text{Vol} = \int_0^h \frac{s^2}{h^2}x^2\,dx = \frac{s^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = ?$$

$$= \frac{s^2}{h^2}\frac{h^3}{3} = \frac{1}{3}s^2h$$

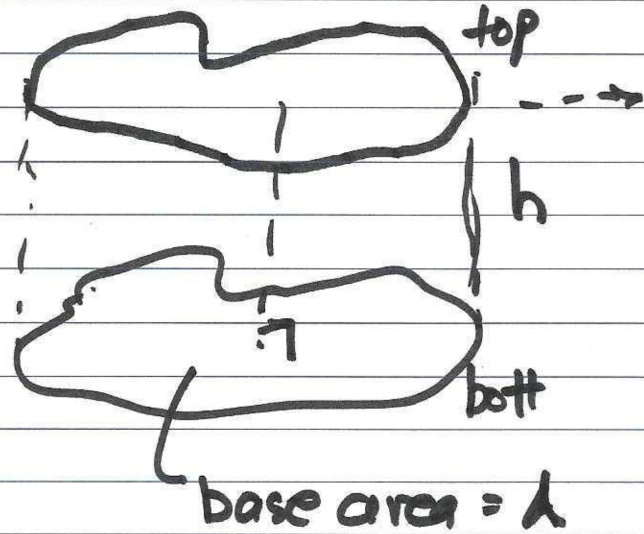
Aside



base with area A

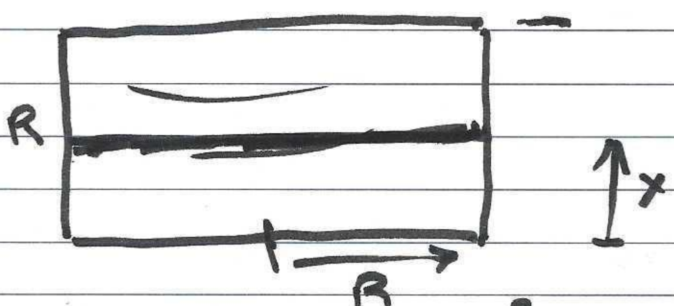
$$\text{Vol} = \frac{1}{3}A \cdot h$$

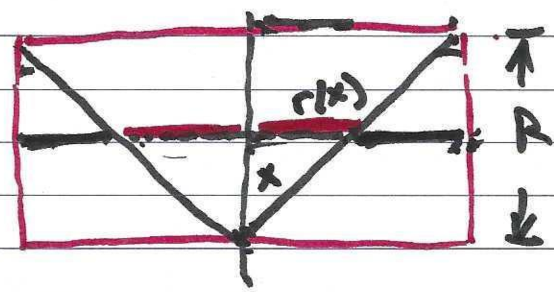
Also:



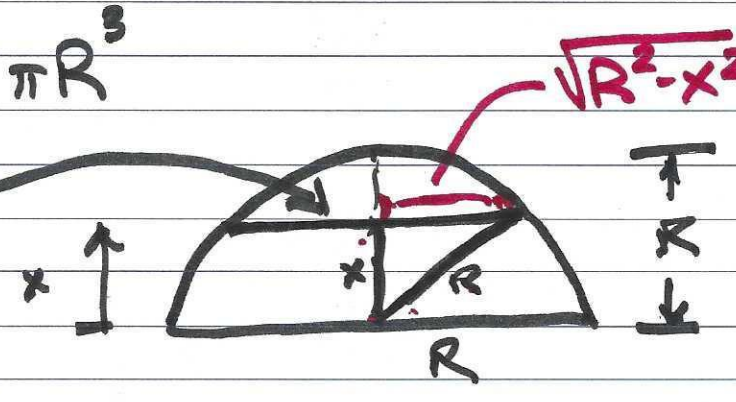
$$\text{Vol} = A \cdot h$$

Archimedes derivation of Volume of a Sphere





$$V = \pi R^2 \cdot R = \pi R^3$$



Area

$$\pi\left(\sqrt{R^2 - x^2}\right)^2 = A(x)$$

5

$$\frac{x}{R} = \frac{r(x)}{R} \Rightarrow r(x) = x$$

Area of cylindrical slice w/ conical slice removed is $\pi R^2 - \pi x^2 = \pi(R^2 - x^2)$

Area slice of hemisphere is $\pi(R^2 - x^2)$

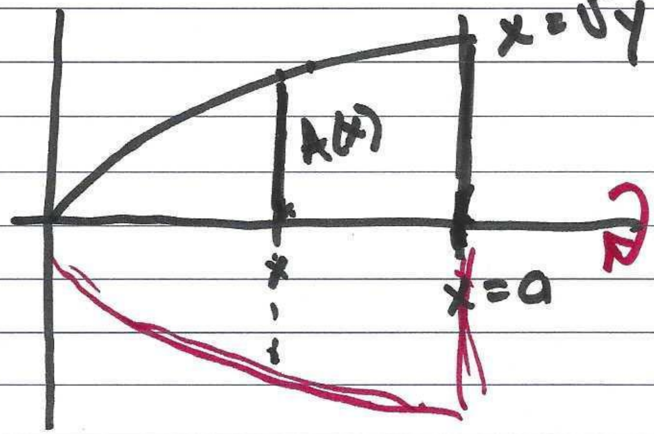
So cyl volume $= \pi R^3$

and conical volume is $\frac{1}{3}\pi R^3$

So hemisphere vol $= \left(1 - \frac{1}{3}\right)\pi R^3 = \frac{2}{3}\pi R^3$

Hence Vol of sphere $= \frac{4}{3}\pi R^3$

⑥

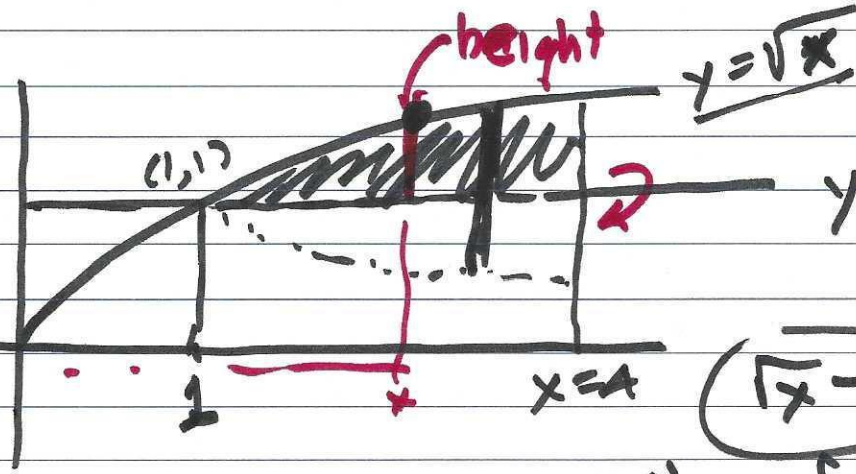


$\underline{x^2 = y}$

slice @ x has radius = $y = x^2$

$$A(x) = \pi (x^2)^2$$

$$\text{Vol} = \int_0^a \pi x^4 \, dx = \pi \left[\frac{x^5}{5} \right]_0^a = \boxed{\frac{\pi a^5}{5}}$$



$\sqrt{x} - 1$ is radius of section

Area of disk @ $x = \pi(\sqrt{x}-1)^2 = \underline{A(x)}$

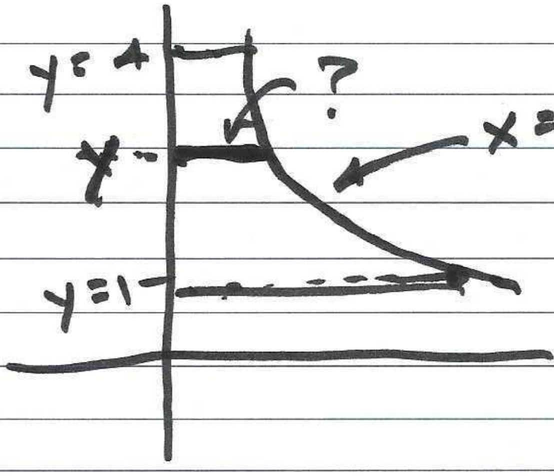
$$\text{Vol} = \int_1^4 \pi(\sqrt{x}-1)^2\,dx =$$

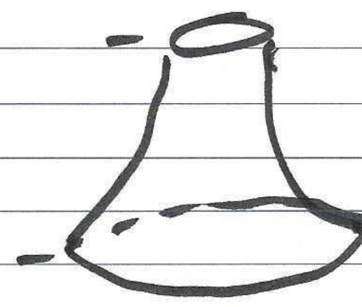
$$= \pi\int_1^4 (x - 2\sqrt{x} + 1)\,dx =$$

$$\pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4$$

$$\pi\left[\frac{15}{2} - \frac{4}{3}\cdot 7 + 3\right] = \underline{\pi\left(\frac{15}{2} - \frac{28}{3} + 3\right)}$$

$xy = 2 \Rightarrow x = \frac{2}{y}$

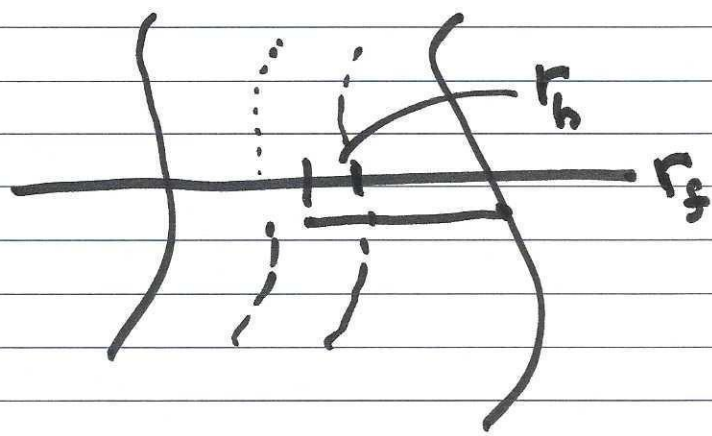




$$A(y) = \pi\left(\frac{2}{y}\right)^2$$

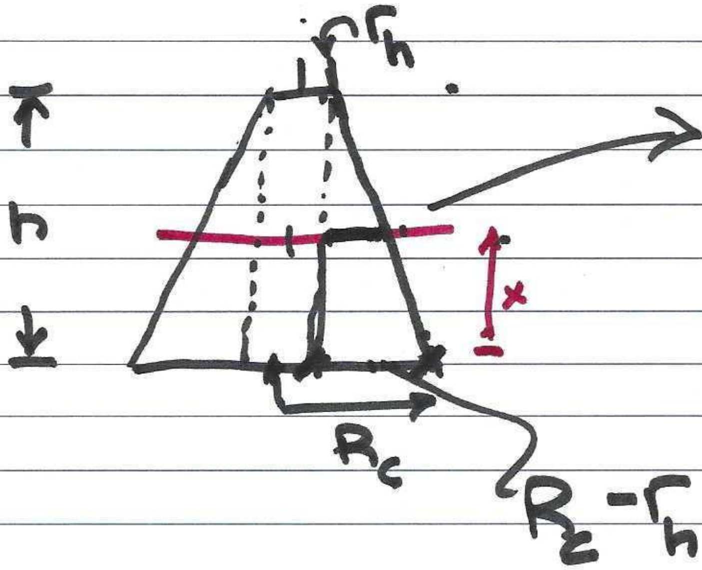
$$\text{vol} = \int_1^4 \pi\left(\frac{4}{y^2}\right)dy = 4\pi\int_1^4 y^{-2}\,dy = 4\pi\left[\frac{-1}{y}\right]_1^4$$

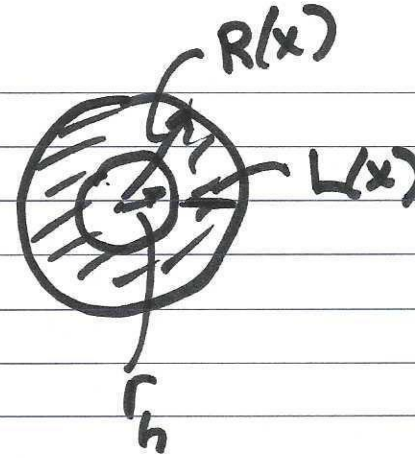
$$= 4\pi\left(-\frac{1}{4}+1\right) = \frac{3}{4}\cdot 4\pi = 3\pi$$



$$\left(\pi r_f^2 - \pi r_h^2\right) = \text{washer area}$$

9





$$\frac{x}{h} =$$

$$\frac{L(x)}{R_c - r_h} \stackrel{?}{=} \frac{x}{h} \Rightarrow L(x) = \frac{(R_c - r_h)x}{h}$$

Cone radius @ x is $L(x) + r_h$

Area of "washer" @ height x is

$$\pi (L(x) + r_h)^2 - \pi r_h^2$$

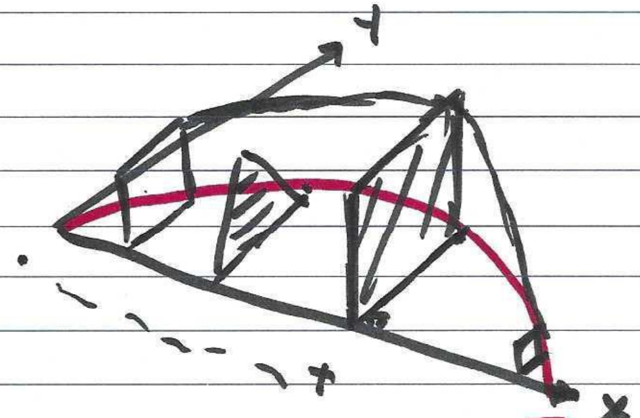
⑩

$$\text{vol} = \pi\int_0^h \left(L^2(x) + 2r_h L(x)\right)dx$$

$$= \pi\int_0^h \left[\left(\frac{R_c - r_h}{h}\right)^2 x^2 + 2r_h\frac{(R_c - r_h)}{h}x\right]dx$$

$$= \pi\left(\frac{R_c - r_h}{h}\right)^2\int_0^h x^2\,dx + 2\pi r_h\left(\frac{R_c - r_h}{h}\right)\int_0^h x\,dx$$

$y = 2\sqrt{\sin x}$

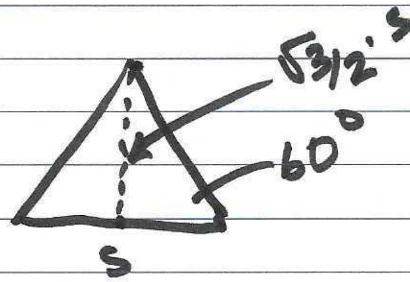


$A(x) = 4\sin x$

(11)

$$V = \int_0^{\pi} 4\sin x\,dx = \Big[-4\cos x\Big]_0^{\pi} = ?$$

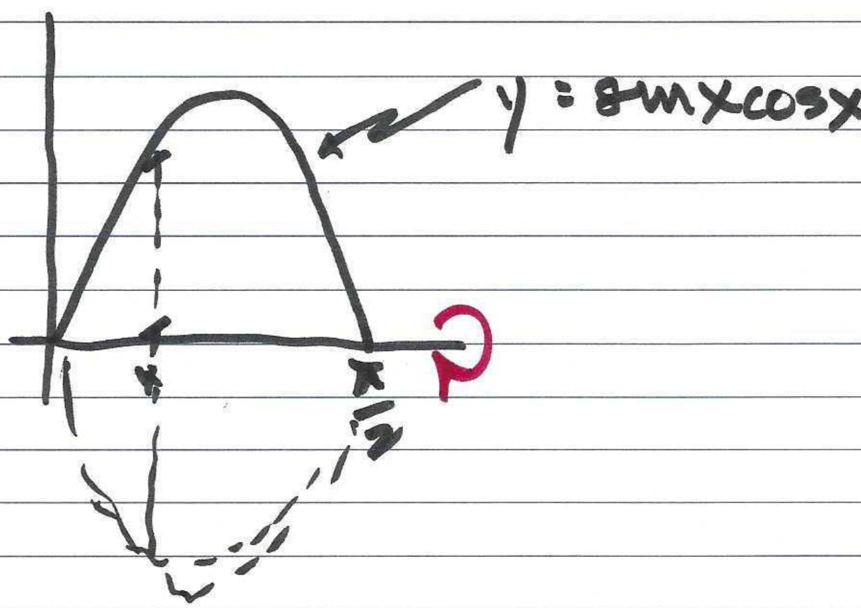
$$4-(-4) = 8$$



$$\frac{1}{2}s \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2 = \text{Area}$$

for Δ Now $A(x) = \frac{\sqrt{3}}{4}\left(4\sin x\right) = \sqrt{3}\cdot\sin x$

$$\text{Vol} = \int_0^{\pi} \sqrt{3}\sin x\,dx = \sqrt{3}\,\underbrace{\Big[-\cos x\Big]_0^{\pi}}_{2} = 2\sqrt{3}$$



$A(x) = \pi \sin^2 x \cos^2 x$

$dV(x) = \pi \sin^2 x \cos^2 x \, dx$

$$V = \pi \int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$$

Use Pythagorean identity

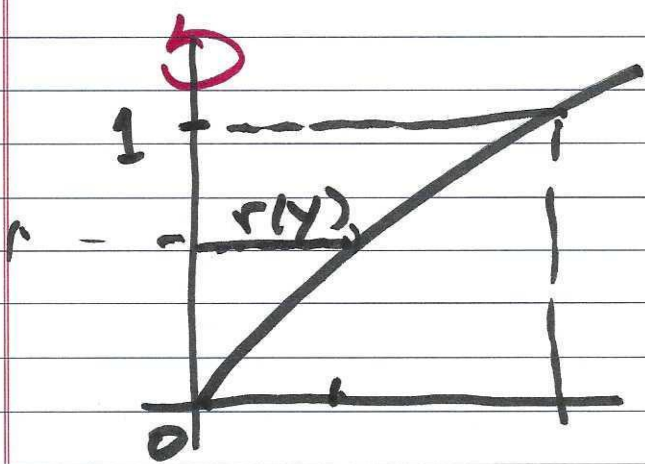
$\sin^2 x \cos^2 x = \sin^2 x (1 - \sin^2 x) =$

$\sin^2 x - \sin^4 x$

(13)

$$V = \pi \int_0^{\pi/2} \left[\sin^2 x - \sin^4 x\right] dx$$

you can look up



$$x = \tan\left(\frac{\pi y}{4}\right)$$

$$A(y) = \pi \tan^2\left(\frac{\pi y}{4}\right)$$

$$\pi \int_0^1 \tan^2\left(\frac{\pi y}{4}\right) dy \stackrel{?}{=}$$

Let $\tan^2\left(\frac{\pi y}{4}\right) = \sec^2\left(\frac{\pi y}{4}\right) - 1$

$$\text{Vol} = \pi \int_0^1 \left[\sec^2\left(\frac{\pi y}{4}\right) - 1\right] dy =$$

(14)

$$\text{Vol} = \pi\left[\frac{4}{\pi}\tan\left(\frac{\pi y}{4}\right) - y\right]_0^1$$

$$= \frac{4\pi}{\pi}\tan\left(\frac{\pi}{4}\right) - \pi$$

$$= 4 - \pi$$