§13.2

Vector integration

Given $\mathbf{r}'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$

Define $\int \mathbf{r}'(t)\,dt = \int x'(t)\,dt\,\hat{i} + \int y'(t)\,dt\,\hat{j} + \int z'(t)\,dt\,\hat{k}$

Ex: $\int_0^{\pi} \left[(\cos t)\,\hat{i} + 1 \cdot \hat{j} - 2t\,\hat{k} \right] dt = ?$

$\left[\sin t\,\hat{i} + t\,\hat{j} - t^2\,\hat{k} \right]_0^{\pi} = 0\,\hat{i} + \pi\,\hat{j} - \pi^2\,\hat{k}$ ✓

Problem: $\mathbf{a}(t) = \mathbf{r}''(t) = (-3\cos t)\,\hat{i} - (3\sin t)\,\hat{j} + 2\hat{k}$

Initial conditions: $\mathbf{r}(0) = (4,0,0) = \underline{4\hat{i} + 0\hat{j} + 0\hat{k}}$

$\mathbf{v}(0) = \mathbf{r}'(0) = 3\hat{j}$ ✓

s
c ←
$-s$
$-c$
s

$$\int^{t}\left[(-3\cos t)\,\hat{i}-(3\sin t)\,\hat{j}+2\hat{k}\right]dt =$$

$$\mathbf{r}'(t)=(-3\sin t)\,\hat{i}+(3\cos t)\,\hat{j}+(2t)\,\hat{k}+\boxed{\mathbf{C}_1}=0$$

$$@\ t=0 \qquad 0 \;+\; 3\hat{j} \;=\; \mathbf{C}_1=\langle 0,3,0\rangle-3\hat{j}=0$$

So $\mathbf{r}'(0)=3\hat{j}$

$$\cancel{3\hat{j}}+\mathbf{C}_1=\cancel{3\hat{j}}$$

Now integrate $\int \mathbf{r}'(t)\,dt$:

$$\mathbf{r}(t)=\int^{t}\left[(-3\sin t)\,\hat{i}+(3\cos t)\,\hat{j}+2t\,\hat{k}\right]dt$$

$$=(3\cos t)\,\hat{i}+(3\sin t)\,\hat{j}+t^2\hat{k}+\mathbf{C}_2$$

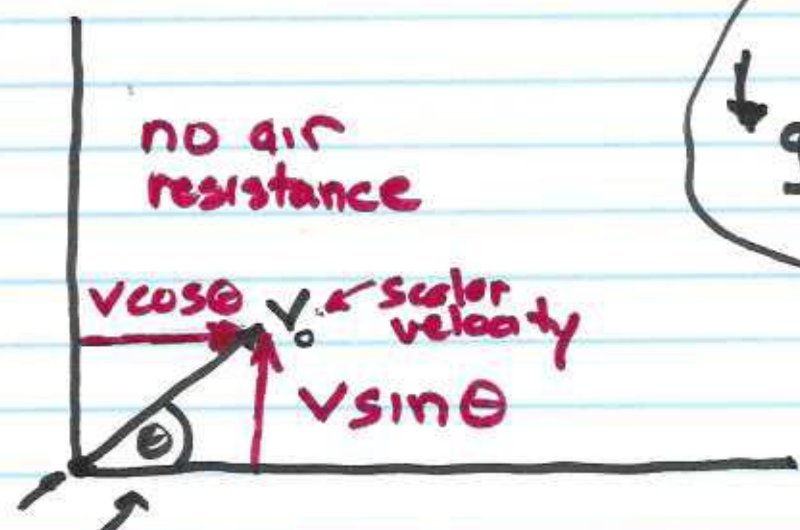
$$\mathbf{r}(0)= \quad 3\hat{i}+0\hat{j}+0\hat{k}+\mathbf{C}_2=4\hat{i}+0\hat{j}+0\hat{k}$$

Claim $\mathbf{C}_2=\hat{i}+0\hat{j}+0\hat{k}$

$$\mathbf{r}(t)=\left((3\cos t)+1\right)\hat{i}+(3\sin t)\,\hat{j}+t^2\hat{k}$$

In $\mathbb{R}^2$

$$g = 9.81 \frac{m}{s^2}$$



launch angle / firing angle / angle of elev'n

Let $\mathbf{r}(t)$ be the path vector for projectile.

(F = ma)

$$\cancel{m}\,\mathbf{r}''(t) = -\cancel{m}\,g\,\hat{j}$$

$$\mathbf{r}''(t) = -g\,\hat{j}$$

$$\mathbf{r}(0) = \mathbf{r}_0$$

$$\mathbf{r}'(0) = v_0\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]$$

(1)

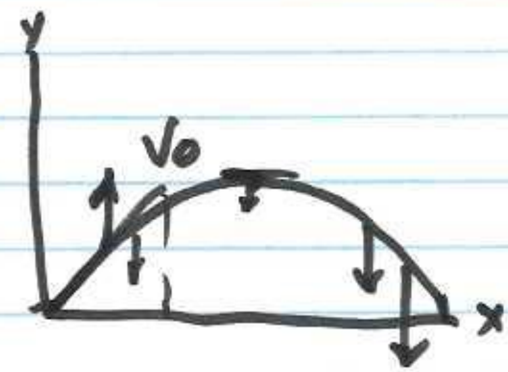
Integrate $\mathbf{r}''(t) = -g\hat{j}$ wrt

$$\int^{t} \mathbf{r}''(t)\,dt = -g\int^{t} \hat{j}\,dt = -g\underset{0}{\underset{\uparrow}{t}}\,\hat{j} + C_0$$

$C_0 = v_0\hat{i}$

So... $\mathbf{r}'(t) =$

$$= v_0\hat{i} - gt\hat{j} \leftarrow$$



$$\int \mathbf{r}'(t)\,dt = v_0 t\,\hat{i} - \frac{gt^2}{2}\hat{j} + C_1$$

@ $t=0$ $\quad v_0\cdot\cancel{0}\,\hat{i} - \cancel{\frac{gt^2}{2}}\hat{j} + C = r_0$

Final result $\mathbf{r}(t) = (v_0\cos\theta)t\,\hat{i} + (v_0\sin\theta)t\,\hat{j}$
$- \frac{1}{2}gt^2\hat{j}$

⑤

$$\mathbf{r}'(10) = \mathbf{v}_0\,\hat{i} - g t \hat{j}$$

$$\downarrow$$

$$= v_0(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) - g t \hat{j}$$

@ t=10 $= \left[(500)(.866) - (9.81)(10)\right]\hat{j}$

$450 - 98 \sim 352$ m/s vert.

Max height $= \dfrac{(v_0 \sin\theta)^2}{2g}$

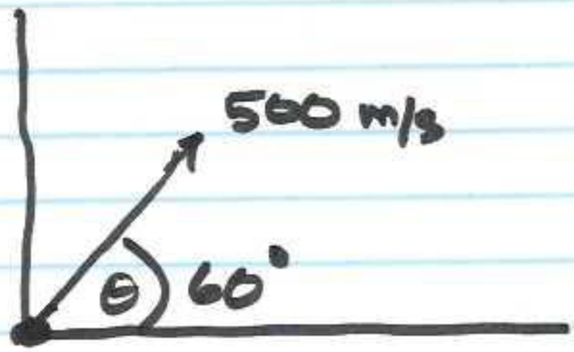
Flight time $= \dfrac{2 v_0 \sin\theta}{g}$

Range $= \dfrac{v_0^2}{g} \sin 2\theta$

Previous case: Max ht $= \dfrac{(500 \cdot .866)^2}{2 \cdot 9.81} = 9556$

$$\mathbf{r}(t) = v_0 \cos\theta\, t\, \hat{i} + \left[v_0 \sin\theta\, t - \tfrac{1}{2} g t^2 \right] \hat{j} + \mathbf{r}_0$$

Example



$\mathbf{r}_0 = 0$

Where is projectile @ $t = 10$ s.?

$$\mathbf{r}(10) = 500\left(\tfrac{1}{2}\right)(10)\, \hat{i} + \left[500(.866)(10) - \tfrac{1}{2}(9.81)(100) \right] \hat{j}$$

$2500\, \hat{i}$

3839.6

Ans. $\mathbf{r}(10) = 2500\, \hat{i} + 3839.6\, \hat{j}$

$$\mathbf{r}'(10) = \left[(v_0 \sin\theta) t - \tfrac{1}{2} g t^2 \right]_{t=10}$$

$4330 - 48 \sim 4282$ m/s.

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flight time $= \dfrac{2(500)(.866)}{9.81} = 88.3$ s.

Range $= \dfrac{(500)^2}{9.81} \cdot (.866) = \underline{22{,}066}$