

late
10/8

①

Integration by Parts (partial integration)

Product Rule in reverse.

If $u(x), v(x)$ are functions then

$$\int \frac{d}{dx} (u(x) \cdot v(x)) dx = \int u'(x) v(x) dx + \int u(x) v'(x) dx$$

$$u(x)v(x) = \int v(x)[u'(x)dx] + \int u(x)[v'(x)dx]$$

$$\underbrace{uv}_{\text{no brain}} = \underbrace{\int v du}_{\text{easy}} + \underbrace{\int u dv}_{\text{difficult}}$$

$$\int \underline{u} dv = uv - \int v du$$

$$\text{Ex 1: } \int \underbrace{x \cos x dx}_{\substack{\uparrow u \quad dv}} = \underbrace{x \sin x}_{\substack{\uparrow u \quad \uparrow v}} - \int \underbrace{\sin x (1) dx}_{\substack{\uparrow v \quad du}}$$

②

So... $\int x \cos x dx = x \sin x + \cos x$

Ex 2: What is $\int \ln x dx = \int \overset{\uparrow}{1} \cdot \underset{\downarrow}{\ln x} dx$

$$\int \overset{u}{\downarrow} \ln x dx = \overset{v \cdot u}{x} \ln x - \int \overset{v \cdot du}{x} \left(\frac{1}{x} \right) dx$$

$$= x \ln x - x$$

Ex 3: ~~$\int x \ln x dx$~~ $\rightarrow$ ~~$\int x^2 \ln x dx$~~

~~$x(x \ln x - x) = \int \ln x dx$~~

$$\int \overset{\uparrow}{x^2} \underset{\downarrow}{\ln x} dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{1}{2} (x^2 \ln x) - \frac{1}{2} \frac{x^2}{2}$$

$$= \frac{x^2}{2} (\ln x - 1)$$

③

$$\int x^4 e^x dx = ?$$

d/dx	$\int dx$
x^4	e^x
$4x^3$	$+ e^x$
$12x^2$	$- e^x$
$24x$	$+ e^x$
24	$- e^x$
0	$+ e^x$

$$e^x x^4 - e^x (4x^3) + e^x (12x^2) - e^x (24x) + 24e^x$$
$$e^x [x^4 - 4x^3 + 12x^2 - 24x + 24] \text{ Ans.}$$

④

$$\int \underbrace{x^2}_{\downarrow} \underbrace{e^x}_{\uparrow} dx = x^2 e^x - \int e^x (2x dx)$$
$$= x^2 e^x - 2 \int x e^x dx \quad \text{Partial \#1}$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\int x^2 e^x dx = x^2 e^x - 2 [x e^x - e^x] \Rightarrow$$
$$= x^2 e^x - 2x e^x + 2e^x$$
$$= e^x [x^2 - 2x + 2]$$

$$\underline{\int e^x \cos x dx}$$

⑤

$$\int t^2 \cos t dt = ?$$

d/dt	dt
t^2	$\cos t$
$2t$	$\sin t$
2	$-\cos t$
0	$-\sin t$

$$\int \frac{dx}{1+x^2} = \arctan x$$

$$t^2 \sin t + 2t \cos t - 2 \sin t \quad \checkmark + C$$

$$\int \arctan y dy = \int \overset{\uparrow}{1} \cdot \underset{\downarrow}{\arctan y} dy \Rightarrow$$

$$y \arctan y - \int \frac{y dy}{1+y^2}$$

$$\text{Let } u = 1+y^2 \text{ so } du = 2y dy$$

$$\Rightarrow y dy = \frac{du}{2}$$

⑥

$$\int \frac{y dy}{1+y^2} = \int \frac{du/2}{u} = \frac{1}{2} \int \frac{du}{u} =$$

$$\frac{1}{2} \ln|u| = \frac{1}{2} \ln|1+y^2| = \ln \sqrt{y^2+1}$$

$$\int \arctan y dy = y \arctan y - \ln \sqrt{y^2+1}$$

$$\int \sin(\ln x) dx \quad \text{let } u = \ln x$$

$$du = \frac{dx}{x}$$

$$\int \sin(u) \cdot x du = \int (\overset{\uparrow}{\sin u}) (\underset{\downarrow}{\ln u}) du$$

$$= (-\cos u)(\ln u) + \int \frac{\cos u}{u} du$$

$$\textcircled{7} \quad u = \sqrt{x} \quad du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2u}$$

$$\int \cos \sqrt{x} dx = ? \quad \int \cos u \cdot (2u du) \Rightarrow$$

$$\begin{aligned} \textcircled{2} \int u \cos u du &= u \sin u - \int \sin u \cdot 1 du \\ &= u \sin u - (-\cos u) \\ &= u \sin u + \cos u \end{aligned}$$

$$\int \cos \sqrt{x} dx = \frac{1}{2} \left(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x} \right)$$

$$\int \sqrt{x} e^{\sqrt{x}} dx = ? \quad \text{let } u = \sqrt{x}$$

$$du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2u}$$

$$\begin{aligned} &\int u e^u (2u du) \\ &= 2 \int u^2 e^u du \end{aligned}$$

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d/du	$\int du$
u^2	e^u
$2u$	e^u
2	e^u
0	e^u

$$= u^2 e^u - 2u e^u + 2e^u$$

$$e^u [u^2 - 2u + 2]$$

$$\int \sqrt{x} e^{\sqrt{x}} dx = 2e^{\sqrt{x}} [x - 2\sqrt{x} + 2]$$

$$\int \sqrt{x} \ln x dx \quad \text{Let } u = \sqrt{x} \quad du = \frac{dx}{2u}$$

$$\ln u^2 = 2 \ln u$$

$$dx = 2u du$$

$$= 4 \int u^2 \ln u du$$

d/du	$\int$
u^2	$\ln u$
$2u$	$u \ln u - u$
2	u^2
0	u^3

complete

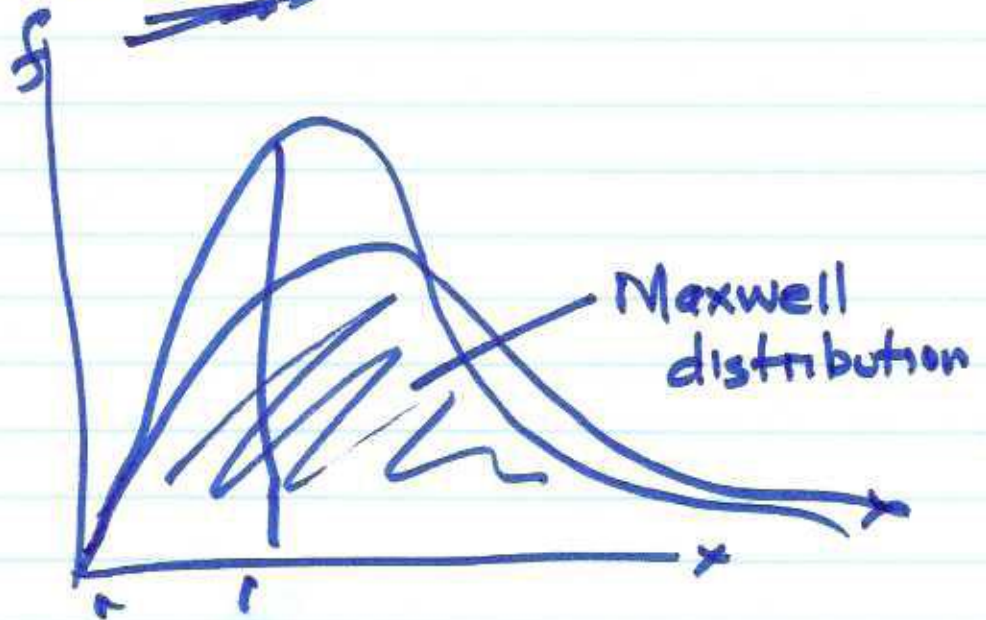
⑨

$$\int x^2 \sin x^3 dx = ? \quad \text{Let } u = x^3$$
$$du = 3x^2 dx$$

$$\int \sin u \frac{du}{3} = -\frac{\cos u}{3} = \boxed{-\frac{\cos x^3}{3}}$$

$$\int x (\ln x)^2 dx = ?$$

Aside: $f(x) = \underline{x^2 e^{-x}}$



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$$u = \ln x \quad x = e^u \quad dx = e^u du$$

$$\int e^u \cdot u^2 \cdot e^u du = \int e^{2u} \cdot u^2 du$$

d/du	$\int du$
u^2	e^{2u}
$2u$	$e^{2u}/2$
2	$e^{2u}/4$
0	$e^{2u}/8$

$$\frac{u^2 \cdot e^{2u}}{2} - \cancel{2u} \cdot \frac{e^{2u}}{\cancel{2}} + \frac{2e^{2u}}{8}$$

$$\frac{e^{2u}}{2} \left[u^2 - u + \frac{1}{2} \right] =$$

(11)

$$= \frac{x}{2} \left[(\ln x)^2 - (\ln x) + \frac{1}{2} \right]$$

$$\int \sin^m \theta \cos^n \theta d\theta \quad \cos^2 \theta = 1 - \sin^2 \theta$$

$$\underline{\cos \theta d\theta = d(\sin \theta)}$$

① m odd

② n odd

③ neither odd

$$\frac{d \sin \theta}{d\theta} = \cos \theta$$

$$d(\sin \theta) = \cos \theta d\theta$$

Ex of ② $\int \sin^7 x \cos^5 x dx$

$$= \int \sin^6 x \cos^4 x d(\sin x)$$

$$= \int \sin^6 x (1 - \sin^2 x)^2 d(\sin x)$$

$$= \int \sin^6 x [1 - 2\sin^2 x + \sin^4 x] d(\sin x)$$

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$$= \int \sin^7 x d(\sin x) - 2 \int \sin^9 x d(\sin x) + \int \sin^{11} x d(\sin x)$$

$$= \frac{\sin^8 x}{8} - 2 \frac{\sin^{10} x}{10 \cdot 5} + \frac{\sin^{12} x}{12} + C$$

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$x = \frac{\theta}{2}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

(13)

$$\int \tan^4 x dx \quad \begin{aligned} \tan^2 x + 1 &= \sec^2 x \\ \tan^2 x &= \sec^2 x - 1 \end{aligned}$$

$$\int (\sec^2 x - 1)^2 dx$$

$$\int (\sec^4 x - 2 \sec^2 x + 1) dx$$

$$\int \sec^4 x dx - 2 \int \sec^2 x dx + \int dx$$

$\quad \quad \quad -2 \tan x \quad + \quad \downarrow$
 $\quad \quad \quad \quad \quad \quad \quad x$

$$\int (\tan^2 x + 1)^2 dx$$

$\downarrow$

$$\int (\tan^2 x + 1) \underbrace{(\sec^2 x)}_{d(\tan x)} dx$$