

late
10/15
①

8.5

Partial Fraction Decomposition:

How do you integrate $\frac{5x-3}{x^2-2x-3}$?

A rational function is a ratio of two polynomials

$$\frac{5x-3}{\underbrace{x^2-2x-3}_{(x-3)(x+1)}} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$(*) \quad 5x-3 = A(x+1) + B(x-3)$$

$$5x-3 = x[A+B] + 1[A-3B]$$

$$\begin{cases} A+B=5 \\ A-3B=-3 \end{cases}$$

$$\text{Let } x = -1 : -8 = A(0) + B(-4)$$

$$-4B = -8 \Rightarrow B=2 \quad A=3$$

(2)

$$\int \frac{5x-3}{x^2-2x-3} dx = \int \frac{3dx}{x-3} + \int \frac{2dx}{x+1}$$

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$$3 \ln|x-3| + 2 \ln|x+1| + C$$

Polynomials with real coefficients can always be expressed as products of linear & quadratic factors

First step is always to compare degrees of the numerator & denominator

$$\int \frac{x^4 - 5x^3 + 2x^2 - x + 5}{x^2 - 2}$$

(3)

$$\begin{array}{r} x^2 + 4 \\ x^2 - 2 \overline{) x^4 - 5x^3 + 2x^2 - x + 5} \\ \underline{x^4} - x + 5 \\ -2x^2 \\ \underline{4x^2 - x + 5} \\ 4x^2 - 8 \\ \underline{0} \quad \underline{-x + 13} \end{array}$$

$$x^2 + 4 + \frac{-x + 13}{x^2 - 2}$$

$$\int - = \int x^2 dx + \int 4 dx - \int \frac{x - 13}{x^2 - 2} dx$$

If fraction has linear terms to various powers, then the decomposition needs to have a constant (TBD) over each power of each distinct term.

(4)

Ex. $\int \frac{3x+5}{(x-1)^2(x+1)^3} =$ ↲

$$\frac{3x+5}{(x-1)^2(x+1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

$$3x+5 = \frac{(x^2-1)(x+1)^3}{(x-1)(x+1)^3} + B(x+1)^3 + C(x-1)^2(x+1)^2 +$$

$$D(x-1)^2(x+1) + E(x-1)^2$$

$$= A[x^4 + 2x^3 - 2x - 1] + B[x^3 + 3x^2 + 3x + 1] \quad \downarrow$$

$$+ C[x^4 - 2x^2 + 1] + D[x^3 - x^2 - x + 1] +$$

$$E(x^2 - 2x + 1)$$

⑤

$$\begin{aligned} 3x+5 &= x^4[A+C] + x^3[2A+B+D] + \\ &+ x^2[3B-2C-D+E] + \\ &+ x[-2A+3B-D-2E] + \\ &+ 1[-A+B+C+D+E] \end{aligned}$$

$$A+C=0 \quad \Rightarrow \quad A=-C$$

$$2A+B+D=0$$

$$3B-2C-D+E=0$$

$$-2A+3B-D-2E=3$$

$$-A+B+C+D+E=5$$

(11)

$$s^4 + 81 = s^4[A + B] + s^3[\cancel{9A} + C] + s^2[18A + D] + s[9C + E] + 1[81A]$$

$$A = 1$$

$$B = 0$$

$$C = 0$$

$$D = 18$$

$$E = 0$$

$$D + 18 = 0$$

$$D = -18$$

$$9 \int \frac{du}{u^2}$$

$$\int \frac{(s^4 + 81) ds}{s(s^2 + 9)^2} = \int \frac{ds}{s} - \int \frac{18s}{(s^2 + 9)^2}$$

$$\downarrow$$
$$\ln|s|$$

$$\downarrow$$
$$9 \left(-\frac{1}{s^2 + 9} \right)$$

$$u = s^2 + 9$$
$$du = 2s$$

$$\ln|s| - \frac{9}{s^2 + 9}$$

Better Ex: $\int \frac{-2x+4}{(x^2+1)(x-1)^2} dx$

⑥ $\begin{array}{l} (Ax+B)(x^2-2x+1) \\ Ax^3-2Ax^2+Ax \\ +Bx^2-2Bx+B \end{array}$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$= (Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$$= x^3[A+C] + x^2[-2A+B-C+D] +$$

$$x[A+B+C] + 1[B-C+D]$$

$$A-2B+C$$

⑦

$$x=1 \rightarrow 2 = D \cdot 2 \Rightarrow D=1$$

$$A = -C$$

$$-2Bx = -2x \Rightarrow B=1$$

$$1 - C + 1 = 4 \Rightarrow C = -2$$

$$A = +2$$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2} dx = \int \frac{2x+1 dx}{x^2+1} - \int \frac{2 dx}{(x-1)} + \int \frac{1 dx}{(x-1)^2}$$

$\downarrow$
 $2 \ln|x-1|$

$\downarrow$
 $U = x-1$
 $dU = dx$
 $-\frac{1}{x-1}$

(8)

$$\int \frac{(2x+1)dx}{x^2+1} = 2 \int \frac{x dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

$$u = x^2 + 1$$
$$du = 2x dx$$

$$\ln|x^2+1|$$

$$\downarrow$$
$$\arctan x$$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2} = \ln|x^2+1| + \arctan x +$$
$$2 \ln|x-1| - \frac{1}{x-1}$$

$$\int \frac{3t^2 + t + 4}{t^2 + t}$$

⑨

$$\int \frac{3t^2 + t + 4}{t^2 + t} = 3 + \frac{-2t+4}{t(t+1)}$$

$$t^2 + t \overline{) 3t^2 + t + 4}$$

$$\underline{3t^2 + 3t}$$

$$-2t + 4$$

$$\int 3dt + \int \frac{-2t+4}{t(t+1)}$$

$$\int \frac{-2t+4}{t(t+1)} dt = \int \frac{A}{t} + \int \frac{B}{t+1}$$

$$-2t+4 = A(t+1) + Bt$$

$$= t[A+B] + 1[A]$$

$$A+B = -2$$

$$B = -6$$

$$A=4 \quad B=-6$$

10

$$\int 3dt + \int \frac{4dt}{t} - \int \frac{6dt}{t+1}$$

↓

$$3t + 4 \ln|t| - 6 \ln|t+1| + C$$

26) $\int \frac{s^4 + 81}{s(s^2 + 9)^2} ds = ?$

$$\frac{s^4 + 81}{s(s^2 + 9)^2} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9} + \frac{Ds + E}{(s^2 + 9)^2}$$

$$A(s^2 + 9)^2 + (Bs + C)s \overset{s^2 + 9}{(s^2 + 9)} + (Ds + E)s$$

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$$A[s^4 + 18s^2 + 81] + [Bs^4 + 9Bs^{\frac{2}{s}} + Cs^3 + 9Cs] + [Ds^2 + Es]$$