6PANZ

early
9/22
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Identities and Derivatives of Hyperbolic Functions

Recall $\sinh x := \dfrac{e^x - e^{-x}}{2}$ (half difference)

$\sinh 0 = 0$
$\cosh 0 = 1$

$\cosh x := \dfrac{e^x + e^{-x}}{2}$ (average)

all others built from these by analogy with trig functions:

$\tanh x := \dfrac{\sinh x}{\cosh x}$ defined for all x

$\coth x := \dfrac{\cosh x}{\sinh x}$ not defined @ $x=0$

$\operatorname{sech} x := \dfrac{1}{\cosh x}$ defined for all x

$\operatorname{csch} x := \dfrac{1}{\sinh x}$ not defined @ $x=0$

1) $$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{1}{2}(e^{x} - e^{-x})\right)$$

$$= \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$$

2) $$\frac{d}{dx}(\cosh x) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh x$$

3) $$\frac{d}{dx}(\operatorname{sech} x) = \left(\frac{1}{\cosh x}\right)' =$$

$$= \frac{\cosh x\,(0) - (1)(\sinh x)}{\cosh^2 x}$$

$$= \frac{-\sinh x}{\cosh^2 x} = \boxed{-\tanh x \cdot \operatorname{sech} x}$$

4) $$\frac{d}{dx}(\tanh x) = ? \quad \left(\frac{\sinh x}{\cosh x}\right)' =$$

$$\frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x$$

Let $u = u(x)$ $\frac{d}{dx}(\tanh u)$

$$\frac{d}{dx}\left(\tanh(e^x)\right) \stackrel{?}{=}$$

$\frac{d}{dx}$ ([redacted] tanh [redacted]) =

$\operatorname{sech}^2 x \cdot$ [redacted] $'$

$(a+b)^2 = a^2 + 2ab + b^2$

$\sinh^2 x + 1 = \cosh^2 x$ | $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{(e^x - e^{-x})}{2}\right)^2 + 1 \stackrel{?}{=} \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$\frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) + 1 \stackrel{?}{=} \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$$

$$e^{2x} - 2 + e^{-2x} + 4 \stackrel{?}{=} e^{2x} + 2 + e^{-2x}$$

F2

(*) $\sinh(2x) \overset{?}{=} 2\sinh x \cosh x$

Trig analog: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\frac{e^{2x}-e^{-2x}}{\cancel{2}} = 2\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$e^{2x}-e^{-2x} = (e^{x}-e^{-x})(e^{x}+e^{-x})$$

$$e^{2x} - 1 + 1 - e^{-2x}$$

$$e^{2x} - e^{-2x}$$ ✓

When you want derivative of an "inverse" function, do this:

$$\left[f\left(f^{-1}(x)\right)\right]' = f'\left(f^{-1}(x)\right) \cdot \left[f^{-1}(x)\right]'$$

$$\downarrow$$

$$x' = 1$$

$$\left[f'(x)\right]' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Find: $\frac{d}{dx}\left(\cosh^{-1} x\right) =$

$y = \sinh x$

Express x as function of y

$$y = \frac{e^{x} - e^{-x}}{2}$$

$$2y = e^{x} - e^{-x} \Rightarrow e^{x} - 2y - e^{-x} = 0$$

Multiply by e^{x}: $e^{2x} - 2ye^{x} - 1 = 0$

$$(e^{x})^2 - 2y(e^{x}) - 1 = 0$$

$$e^{x} = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

$$e^{x} = y \pm \sqrt{y^2 + 1}$$

$$x = \ln\left(y + \sqrt{y^2 + 1}\right)$$

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\frac{e^{x+y} - e^{-(x+y)}}{2} \stackrel{?}{=}$$

$$\frac{e^{x} - e^{-x}}{\cancel{2}} \cdot \frac{e^{y} + e^{-y}}{\cancel{2}} + \frac{e^{x} + e^{-x}}{\cancel{2}} \cdot \frac{e^{y} - e^{-y}}{\cancel{2}}$$

$$\underbrace{\sinh x \quad \cosh y}\quad \cosh x$$

$$e^{x}e^{y} + \cancel{e^{x}e^{-y}} - \cancel{e^{-x}e^{y}} - e^{-x}e^{-y} +$$

$$e^{x}e^{y} - \cancel{e^{x}e^{-y}} + \cancel{e^{-x}e^{y}} - e^{-x}e^{-y}$$

$$2(e^{x}e^{y}) - 2(e^{-x}e^{-y})$$

$$2\cdot\frac{e^{x}e^{y} - e^{-x-y}}{2} \stackrel{?}{=} e^{x}e^{y} - e^{-x}e^{-y} \checkmark$$

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We say $f(x)$ grows faster than $g(x)$ as $x \to \infty$ if:

$$\lim_{x \to \infty} \left| \frac{g(x)}{f(x)} \right| = 0$$

$\frac{\infty}{\infty} = ?$

Compare: x^2 & 2^x

Look @ $\lim_{x \to \infty} \left| \frac{x^2}{2^x} \right| = ?$

$$\lim_{x \to \infty} \left| \frac{2x}{2^x \ln 2} \right|$$

$$\lim_{x \to \infty} \left| \frac{2}{(\ln 2)^2 \cdot 2^x} \right| \rightsquigarrow 0$$

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$f(x)$ & $g(x)$ grow @ same rate as $x \to \infty$

$$\lim_{x\to\infty} \left| \frac{f(x)}{g(x)} \right| = M, \text{ some constant } M \neq 0$$

Claim: $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow

at same rate as $x \to \infty$

Look @ $\displaystyle\lim_{x\to\infty} \left| \frac{\sqrt{x^2+5}}{x} \right| = \lim_{x\to\infty} \left| \sqrt{1+\frac{5}{x^2}} \right| = \underline{1}$

Look @ $\displaystyle\lim_{x\to\infty} \left| \frac{(2\sqrt{x}-1)^2}{x} \right| = \lim_{x\to\infty} \left| \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 \right| = ?$

$\downarrow$

$\left(2 - \frac{1}{\sqrt{x}}\right)^2 \rightsquigarrow 4$