

8BJAP

early
10/13
①

8.3 Loose Ends

$$\text{Ex 1 } \int \tan^2 x \, dx = ? \tan x - x + C$$

$$\int (\sec^2 x - 1) \, dx = \int \sec^2 x \, dx - \int dx$$

$\downarrow \qquad \qquad \downarrow$
 $\tan x - x$

$$\text{Ex 2 } \int \tan^4 x \, dx = ?$$

$$\int \tan^2 x (\sec^2 x - 1) \, dx = ?$$

$$\int \tan^2 x \underbrace{\sec^2 x \, dx}_{d(\tan x)} - \int \tan^2 x \, dx$$

$\downarrow$

$$\frac{\tan^3 x}{3} - (\tan x - x + C)$$

$$\frac{1}{3} \tan^3 x - \tan x + x + C$$

(2)

$$\int \sec^3 x dx = ? -$$

$$\int \underbrace{\sec x}_u \underbrace{\sec^2 x dx}_{dv} =$$

$$u = \sec x$$

$$dv = \sec^2 x dx$$

$$v = \tan x$$

$$du = \sec x \tan x$$

$$\int u dv = uv - \int v du$$

$$\hookrightarrow \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx$$

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$

③

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What is $\int \sec x dx \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$

Aside: $\int \sec x = \ln |\sec x + \tan x|$

Finally:

$$\star \int \sec^3 x dx = \frac{1}{2} \left[\sec x \tan x + \ln |\sec x + \tan x| \right]$$

$$\int \sin nx \cdot \cos mx \, dx$$

Recall: $\sin(nx+mx) = \sin nx \cos mx + \cos nx \sin mx$

$$\sin(nx-mx) = \sin nx \cos mx - \cos nx \sin mx$$

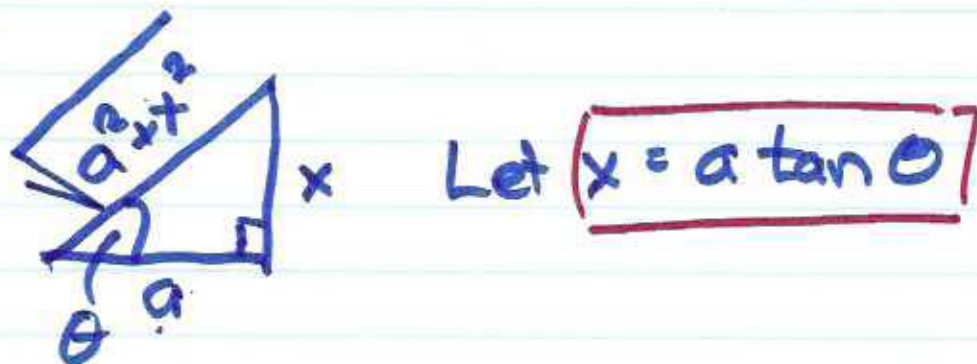
$$2 \sin nx \cos mx$$

$$\frac{1}{2} \left[\sin(nx+mx) - \sin(nx-mx) \right] = \sin nx \cos mx$$

$$\frac{1}{2} \sin(n+m)x - \frac{1}{2} \sin(n-m)x =$$

④

Trig. Substitutions in Integrals



then $\sqrt{a^2 + x^2} = a \sec \theta$

$dx = a \sec^2 \theta d\theta$

Ex: $\int \frac{dx}{\sqrt{4+x^2}}$ let $x = 2 \tan \theta$

$\sqrt{4+x^2} = 2 \sec \theta$

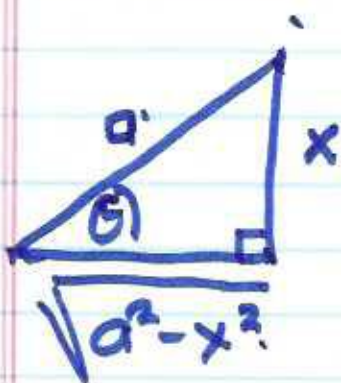
$dx = 2 \sec^2 \theta d\theta$

" $\int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} = \int \sec \theta d\theta =$
 $= \left(\ln |\sec \theta + \tan \theta| \right)$

(5)

$$\theta = \arctan\left(\frac{x}{a}\right)$$

Final answer: $\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$



Let $x = a \sin \theta$

then $\sqrt{a^2 - x^2} = a \cos \theta$

$dx = a \cos \theta d\theta$

Find $\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{a^2 \sin^2 \theta \cancel{a \cos \theta} d\theta}{\cancel{a \cos \theta}}$

$\swarrow a=3$

$= 9 \int \sin^2 \theta d\theta$

⑥

Recall half angle formulas:

$$\begin{array}{l|l} \text{(i)} \sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} & \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2} \\ \text{(ii)} \cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} & \cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2} \\ \text{(iii)} \tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} & \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} \end{array}$$

Continued from ⑤

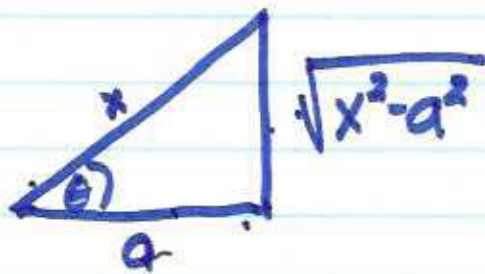
$$\arcsin(\sin 2\theta) = 2\theta$$

$$= 9 \int \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{9}{2} \int d\theta - \frac{9}{2} \int \cos 2\theta d\theta$$

$$= \frac{9\theta}{2} - \frac{9}{2} \frac{\sin 2\theta}{2} = \frac{1}{2} \left[9\theta - \frac{9 \sin 2\theta}{2} \right] \checkmark$$

⑦



$$x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

$$\#99) \int \frac{dx}{x \sqrt{x^2 - 1}} = \int \frac{a \sec \theta \tan \theta d\theta}{a \sec \theta a \tan \theta}$$

$$\underline{a=1} \quad = \int \frac{\cancel{\sec \theta} \cancel{\tan \theta} d\theta}{\cancel{\sec \theta} \cancel{\tan \theta}} = \int d\theta = \theta$$

$= \arcsin x$

(8)

$$49) \int \sqrt{8-2x-x^2} dx$$

$$-(8-2x-x^2) = x^2 + 2x - 8 = (x+1)^2 - 9$$

$$\sqrt{9-(x+1)^2}$$

$$\int \sqrt{9-(x+1)^2} dx \rightsquigarrow \int \sqrt{9-y^2} dy$$

$$y = x+1 \quad \text{Let } y = 3 \sin \theta$$

$$dy = 3 \cos \theta d\theta$$

$$\sqrt{9-y^2} = 3 \cos \theta$$

$$\int (3 \cos \theta)(3 \cos \theta) d\theta = 9 \int \cos^2 \theta d\theta$$

①

$$\text{Note: } \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$= 9 \int \frac{1 + \cos 2\theta}{2} d\theta = \checkmark$$

$$\int \frac{dx}{\sqrt{25x^2 - \frac{4 \cdot 25}{25}}}$$

$$\sqrt{25\left(x^2 - \frac{4}{25}\right)} = 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

$$\frac{1}{5} \int \frac{dx}{\sqrt{x^2 - \left(\frac{2}{5}\right)^2}}$$

$$\text{Let } x = \frac{2}{5} \sec \theta$$

$$dx = \frac{2}{5} \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5} \tan \theta$$

10)

$$= \frac{1}{5} \int \frac{\frac{2}{5} \sec \theta \tan \theta d\theta}{\frac{2}{5} \tan \theta}$$

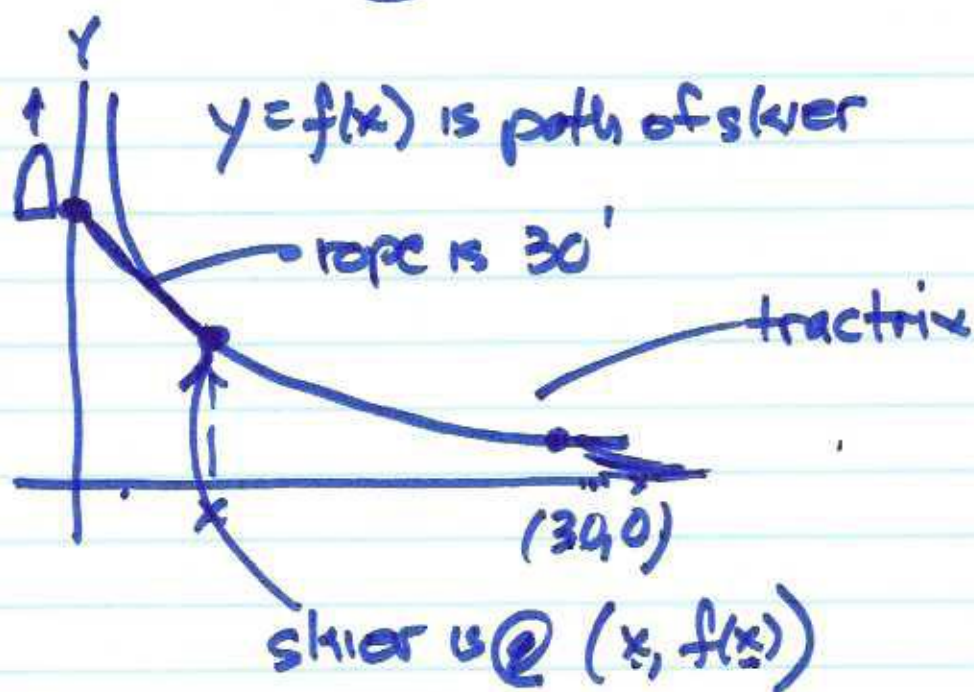
$$= \frac{1}{5} \ln | \sec \theta + \tan \theta |$$

$$\theta = \operatorname{arcsec}\left(\frac{5x}{2}\right)$$

$$= \frac{1}{5} \ln \left| \frac{5x}{2} + \tan \left(\operatorname{arcsec}\left(\frac{5x}{2}\right) \right) \right|$$

$$\underline{f(f^{-1}(x)) = x}$$

(11)



$$f'(x) = \frac{-\sqrt{900 - x^2}}{x} \rightsquigarrow \frac{-\sqrt{a^2 - x^2}}{x}$$

Find $f(x)$ when $f(30) = 0$

This is a $\sqrt{a^2 - x^2}$ case $x = a \sin \theta$

$$dx = a \cos \theta d\theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

$$\left. \begin{array}{l} a = 30 \\ x = 30 \sin \theta \\ dx = 30 \cos \theta d\theta \\ \sqrt{a^2 - x^2} = 30 \cos \theta \end{array} \right\}$$

(12)

$$f(\theta) = \int \frac{-(30 \cos \theta) \cdot \cancel{30} \cos \theta d\theta}{\cancel{30} \sin \theta}$$

$$= -30 \int \frac{\cos^2 \theta}{\sin \theta} d\theta$$

$$= -30 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta \Rightarrow$$

$$= -30 \left[\int \frac{d\theta}{\sin \theta} - \int \sin \theta d\theta \right]$$

$\csc \theta$ →

$$= -30 \left[\ln |\csc \theta + \cot \theta| + \cos \theta \right]$$

$\theta = \arcsin \frac{x}{30}$