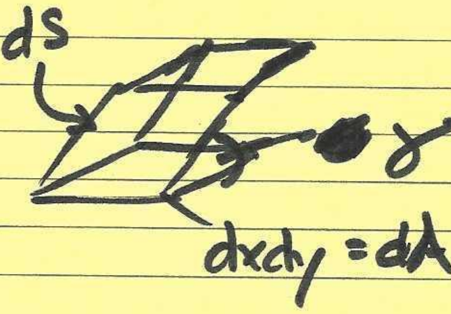
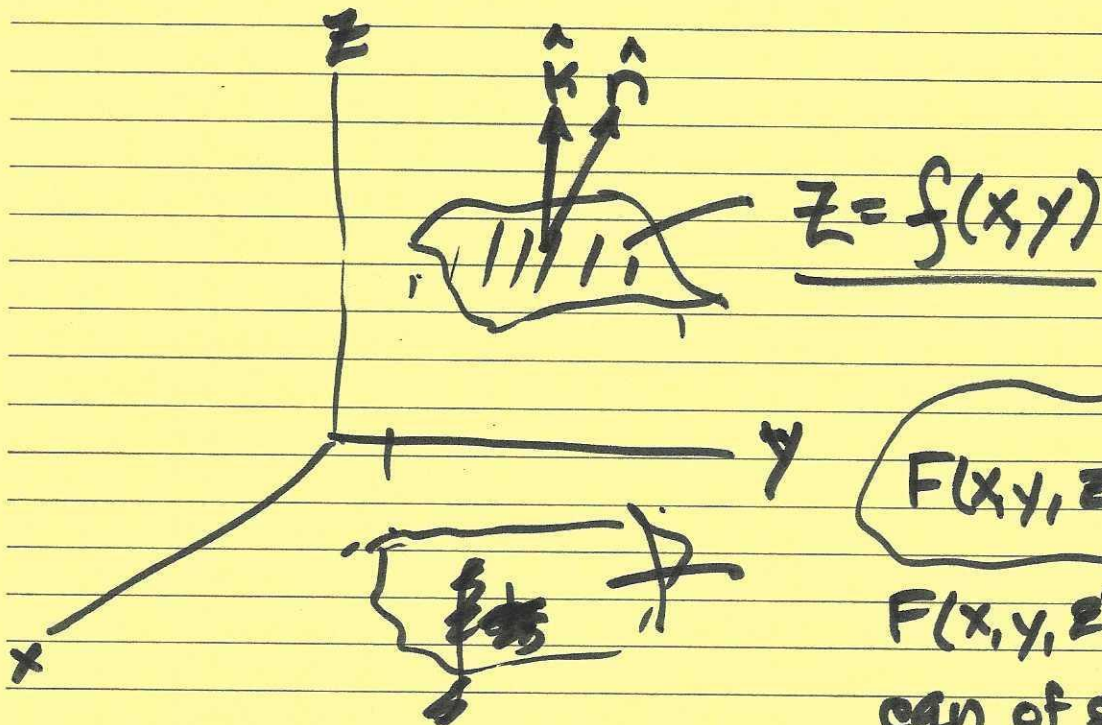
Surface Integrals

$$\iint_S \phi(x,y,z)\,dS$$



$$\frac{dS}{dA} = \sec\gamma = \frac{1}{\cos\gamma}$$



$$F(x,y,z) = z - f(x,y)$$

$F(x,y,z) = 0$ is eqn of surface

$$\sec\gamma = \frac{1}{\hat{n}\cdot\hat{k}} \qquad \frac{\nabla F}{|\nabla F|} = \hat{n}$$

$$\frac{F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}}{\sqrt{F_x^2 + F_y^2 + F_z^2}} = \hat{n}$$

$F_x = -z_x \quad F_y = -z_y \quad F_z = 1$

$$\hat{n}\cdot\hat{k} = \left(\frac{-z_x\,\hat{i} - z_y\,\hat{j} + \hat{k}}{\sqrt{1+z_x^2+z_y^2}}\right)\cdot\hat{k}$$

$$= \frac{1}{\sqrt{1+z_x^2+z_y^2}}$$

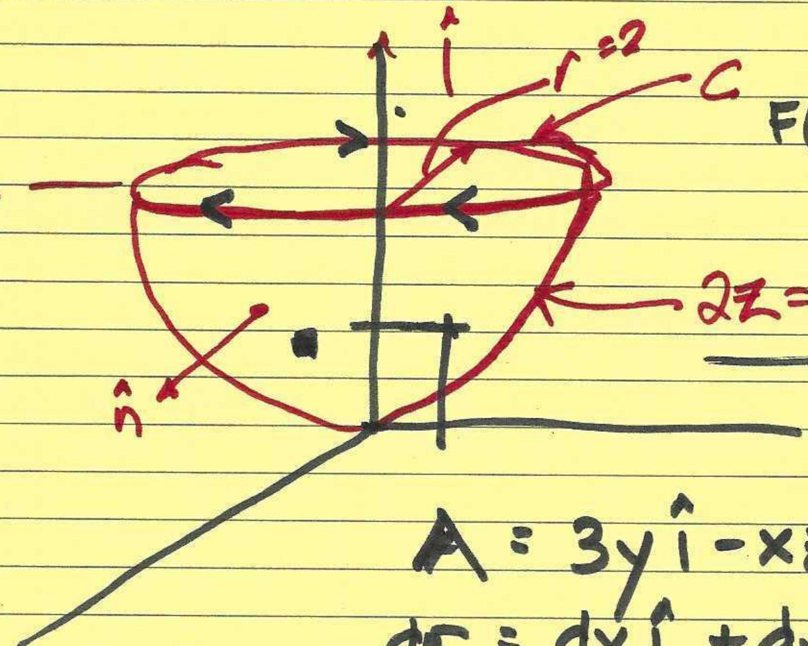
$$\sec\gamma = \frac{1}{\hat{n}\cdot\hat{k}} = \sqrt{1+z_x^2+z_y^2}$$

$$dS = \frac{dA}{\hat{n}\cdot\hat{k}}$$

So $$\iint_S \phi(x,y,z)\,dS = \iint \phi(x,y,z)\sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

Stokes' Th^{m}

$$\iint_S (\nabla \times A) \cdot \hat{n}\, dS = \oint_{C_S} A \cdot d\mathbf{r}$$



$F(x,y,z) = x^2 + y^2 - 2z$

$2z = x^2 + y^2$

$$A = 3y\,\hat{i} - xz\,\hat{j} + yz^2\,\hat{k}$$

$$d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Line integral: $\oint_C A \cdot d\mathbf{r} = \oint_C 3y\,dx - xz\,dy + yz^2 \underbrace{dz}_{0}$

Switch to polar: $x = 2\cos t$, $y = 2\sin t$, $z = 2$

$$\oint_{2\pi}^{0} (6\sin t)(-2\sin t\,dt) - \oint_{2\pi}^{0} (4\cos t)(2\cos t)\,dt$$

$$\int_0^{2\pi} 12\sin^2 t\,dt + \int_0^{2\pi} 8\cos^2 t\,dt$$

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$$\int_0^{2\pi} 8(\sin^2 t + \cos^2 t)\,dt + 4\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt$$

$$\downarrow$$

$$\left[16\pi - 0\right] + 4\left[\pi - 0\right] - \frac{1}{2}\int_0^{2\pi}\cos 2t\,dt$$

$$\downarrow$$

$$-\frac{1}{2}\left[\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$\rightarrow 0$$

$$\oint_C A\cdot dr = 20\pi$$

$$\iint_S (\nabla\times A)\cdot\hat{n}\,dS$$

$$\nabla\times A = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ 3y & -xz & yz^2\end{vmatrix} = 2$$

$$\hat{i}(z^2+x) + \hat{k}(-z-3) = \hat{i}(x+z^2) - \hat{k}(z+3)$$

$$z = \frac{x^2+y^2}{2}$$

$$\boxed{\iint (\nabla \times A)\cdot \hat{n} \, \frac{dA}{|\hat{n}\cdot\hat{k}|}}$$

$$z_x = x \qquad z_y = y$$

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{2x\hat{i}+2y\hat{j}-2}{\sqrt{4x^2+4y^2+4}}$$

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{x\hat{i}+y\hat{j}-1\hat{k}}{\sqrt{1+x^2+y^2}}$$

$$\hat{n}\cdot\hat{k} = \frac{-1}{\sqrt{1+x^2+y^2}} \qquad \frac{1}{|\hat{n}\cdot\hat{k}|} = \boxed{\sqrt{1+x^2+y^2}}$$

$$(\nabla \times A)\cdot\hat{n} = x(z^2+x) + (z+3)$$

$$\iint (xz^2+x^2+z+3)\,dx\,dy$$

$$\boxed{\begin{aligned} x &= 2\cos t \\ y &= 2\sin t \\ z &= 2 \end{aligned}}$$

$$2z = x^2+y^2 = z = \frac{x^2+y^2}{2} \qquad z = 2$$

$$\int_0^{2\pi}\int_0^{2} \left(x\left(\frac{x^2+y^2}{2}\right)^2 + x^2 + \frac{x^2+y^2}{2} + 3 \right) d\rho\, d\theta$$

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$$\int_0^{2\pi}\int_0^{2} (\rho\cos\theta)\left(\frac{\rho^4}{4}\right) + \rho^2\cos^2\theta + \frac{\rho^2}{2} + 3)\,\rho\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\left[\frac{\rho^5\cos\theta}{4} + \rho^2\cos^2\theta + \frac{\rho^2}{2} + 3\right]\rho\, d\rho\, d\theta$$

$$\cancel{= \int_0^{2\pi}\int\left[\frac{\rho^6}{24}\cos\theta\right.}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_0^{2\pi}\left[\frac{\rho^7}{28}\cos\theta + \frac{\rho^4}{4}\cos^2\theta + \frac{\rho^4}{8} + \frac{3\rho^2}{2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[\frac{128}{28}\cos\theta + 4\cos^2\theta + 2 + 6\right] d\theta$$

$$= \left[\underbrace{\frac{128}{28}(+\sin\theta)}_{0} + \boxed{\underbrace{4\left(\frac{1+\cos 2\theta}{2}\right)\theta}_{2}} + \underbrace{8\theta}_{16\pi}\right]_0^{2\pi}$$

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$$2\int(1+\cos\theta)\,d\theta = \Big[2\theta + 2\sin\theta\Big]_0^{2\pi}$$

$$4\pi + 0$$

$$\iint (x^2+y^2)\,dS \text{ where}$$

S is $z^2 = 3(x^2+y^2)$ $0 \le z \le 3$

<u>cone</u>