DUIS / Feynman Integrals

1) $\int_0^\infty \frac{\sin x}{x}\,dx$

2) $\int_0^\infty e^{-x^2}\,dx$

3) $\int_0^\infty \frac{1-\cos(xy)}{x}\,dx$

4) $\int_0^1 \frac{x-1}{\ln x}\,dx$

5) $\int_0^\infty \sin x^2\,dx$

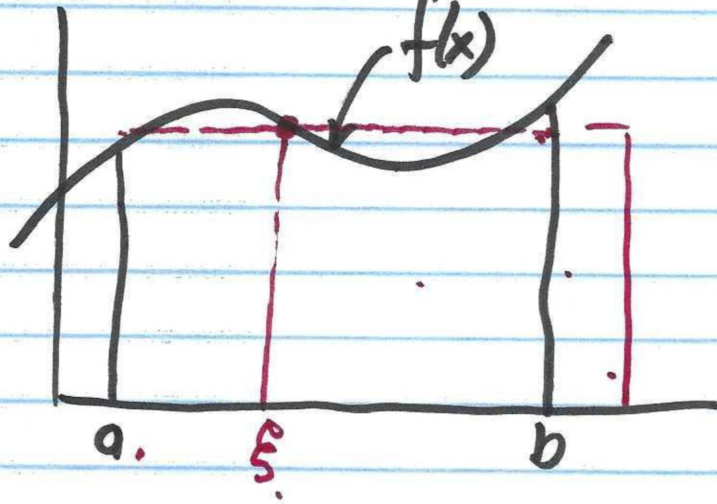
6) $\int_0^\infty \frac{\sin x^2}{x^2(x^2+1)}\,dx$

7) $\int_0^{\pi/2} x\cot x\,dx$

8) $\int_0^{\pi/2} x\sqrt{\tan x}\,dx$

9) $\int \frac{1}{(1+x^2)^2}\,dx$

10) $\int_{-\infty}^\infty \left(\frac{1-\cos x}{x^2}\right)^2 dx$



$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

1st Mean Value Thm

Leibnitz' Th^{m} DULS

Suppose $\phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$

$$\phi'(\alpha) = \lim_{\Delta\alpha \to 0} \frac{\phi(\alpha+\Delta\alpha) - \phi(\alpha)}{\Delta\alpha}$$

$$= \underbrace{\int_{u_1(\alpha+\Delta\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx} - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)\,dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)\,dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx \; -$$

$\int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha$ [redacted] $)\,dx$

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$$\int_{U_1(\alpha)}^{U_2(\alpha)} \left[\frac{f(x,\alpha+\Delta\alpha)-f(x,\alpha)}{\Delta\alpha}\right] dx + \int_{U_2(\alpha)}^{U_2(\alpha+\Delta\alpha)} \frac{f(x,\alpha+\Delta\alpha)\,dx}{\Delta\alpha} -$$

$$\int_{U_1(\alpha)}^{U_1(\alpha+\Delta\alpha)} \frac{f(x,\alpha+\Delta\alpha)\,dx}{\Delta\alpha} =$$

$$\int_{U_1(\alpha)}^{U_2(\alpha)} \left[\frac{f(x,\alpha+\Delta\alpha)-f(x,\alpha)}{\Delta\alpha}\right] dx +$$

$$\left[\frac{U_2(\alpha+\Delta\alpha)-U_2(\alpha)}{\Delta\alpha}\right] f(\hat{x},\alpha+\Delta\alpha) -$$

$$\left[\frac{U_1(\alpha+\Delta\alpha)-U_1(\alpha)}{\Delta\alpha}\right]\cdot f(\hat{x},\alpha+\Delta\alpha) =$$

apply limit

$$\int_{U_1(\alpha)}^{U_2(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx + U_2'(\alpha)\, f(x,\alpha) - U_1'(\alpha)\, f(x,\alpha)$$

(A)

$$\text{So } \phi'(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} \frac{\partial f}{\partial \alpha} f(x,\alpha)\,dx + u_2'(\alpha) f(x,\alpha)$$

$$- u_1'(\alpha) f(x,\alpha) \;\star$$

$$\int \frac{dx}{(1+x^2)^2} = ? \qquad (\alpha^2+x^2)^{-1}$$

$$I(\alpha) = \int \frac{dx}{\alpha^2+x^2} = \boxed{\frac{1}{\alpha}\arctan\frac{x}{\alpha} + C}$$

$$I'(\alpha) = \int \frac{-dx}{(\alpha^2+x^2)^2} \cdot 2\alpha = \int \frac{-2\alpha\,dx}{(\alpha^2+x^2)^2}$$

$$I'(\alpha) = -\frac{1}{\alpha^2}\arctan\frac{x}{\alpha} + \frac{1}{\alpha}\frac{1}{\alpha^2+x^2}\cdot\left(-\frac{x}{\alpha^2}\right)$$

$$-2\int \frac{dx}{(1+x^2)^2} = -\arctan x - \frac{x}{1+x^2}$$

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + C$$

$$\int_0^\infty \frac{\sin x}{x}\,dx \xrightarrow{\text{try}} \int_0^\infty \frac{\sin(\alpha x)}{x}\,dx = I(\alpha)$$

$$I'(\alpha) = \int_0^\infty \frac{\cancel{x}\cos(\alpha x)}{\cancel{x}}\,dx = \int_0^\infty \cos(\alpha x)\,dx$$

??

$$= \frac{\sin \alpha x}{\alpha}\Bigg|_0^\infty = x\cos\alpha x - x$$

$$x(\cos\alpha x - 1)$$

@ ∞ $\dfrac{\cancel{x}\cos\alpha x}{1}$ @ $0 = x$

So... $\boxed{x(\cos x - 1)}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdot$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - + \cdots$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2}\,dx \longrightarrow I(\alpha) = \int_0^\infty e^{-\alpha x^2}\,dx$$

$$I'(\alpha) = -\int_0^\infty (2\alpha x^2)\, e^{-\alpha^2 x^2}\,dx$$

$$I'(\alpha) = -\int_0^\infty 2\alpha x^2 e^{-\alpha^2 x^2}\,dx$$

$e^{-\alpha x^2}$	$2\alpha x^2$

⑦

$$\int_0^\infty \sin x^2 \, dx$$

$$\int_0^\infty \sin \alpha^2 x^2 \, dx \xrightarrow{\partial/\partial\alpha} \int_0^\infty 2\alpha x^2 \sin \alpha^2 x^2 \, dx = I'(\alpha)$$

$$I(\alpha) = \int_\alpha \int_0^\infty 2\alpha x^2 \sin \alpha^2 x^2 \, dx \, d\alpha$$

$$2\int_0^\infty x^2 \int_\alpha \alpha \sin \alpha^2 x^2 \, d\alpha \, dx$$

$$u = \alpha^2 x^2$$

$$du = 2\alpha x^2 d\alpha$$

$$\int_0^\infty x^2 \int_\alpha \sin u \, du \, dx = \int_0^\infty x^2(-\cos\alpha^2 x^2) \, dx$$