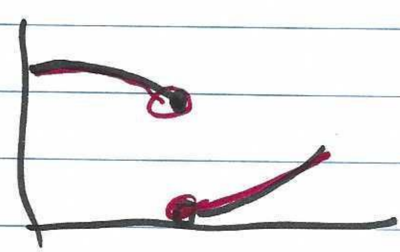
$$\begin{array}{ccc} \text{Diff Eqn (hard)} & \longrightarrow & \text{Solution} \\ \mathcal{L}\downarrow & & \uparrow \mathcal{L}^{-1} \\ \text{algebraic eqn} & \xrightarrow[\text{(maybe easier)}]{\text{solve}} & \text{Alg Sol'n} \end{array}$$

$\mathcal{L}$: time → frequency

$$\mathcal{L}(f(t)) = \int_0^\infty \underbrace{e^{-st}}_{\text{kernel}} f(t)\,dt = F(s)$$

$f(t)$: (i) must be piecewise continuous
(ii) $|f(t)| < Me^{\gamma t}$ some $M>0$ $\gamma \in \mathbb{R}$

Can find X-form if f(t) obeys these conditions



1. Let $f(t) = 1$ $\quad \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$

2. Let $f(t) = t$ $\quad \int_0^\infty e^{-st}\,t\,dt = ?$

$$= \left[\frac{e^{-st}}{s}\cdot t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{s}\,dt = \frac{-1}{s}\int_0^\infty e^{-st}\,dt = +\frac{1}{s^2}(1)$$

$\downarrow$ 0

$= \frac{1}{s^2}$

3. Let $f(t) = t^n$ $\quad \int_0^\infty e^{-st}\,t^n\,dt = ?$

$$\left[\frac{e^{-st}t^n}{-s}\right]_0^\infty - \left(\frac{n}{-s}\right)\underline{\int_0^\infty e^{-st}\,t^{n-1}\,dt}$$

$\downarrow$ 0

$= \frac{1}{s}\mathcal{L}(t^{n-1})$

$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

③

④ Let $f(t) = e^{at}$ $\int_0^\infty e^{-st} e^{at} dt =$

$$\int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty =$$

$$= \frac{1}{s-a}$$

⑤ Let $f(t) = \sin(at), \cos(at)$

Recall $e^{iat} = \cos at + i \sin at$ (Euler)

$$\frac{1}{s-ia} \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} =$$

$$\frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

C^∞, C^ω

Let: $f'(t)$ $\quad \int_0^\infty e^{-st} f'(t)\,dt =$

$$\left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

$$0 - f(0) + s \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}(f(t))}$$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

Let: $f''(t)$ $\quad \int_0^\infty e^{-st} f''(t)\,dt$

$$\left[e^{-st} f'(t)\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= 0 - f'(0) + s\left[sF(s) - f(0)\right]$$

$$= s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}(f'''(t)) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

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1. $y'' + y = t \quad y(0) = 1, \; y'(0) = -2$

$$\left(s^2 Y - s(1) + 2\right) + Y = \frac{1}{s^2}$$

$$s^2 Y + Y - s + 2 = \frac{1}{s^2}$$

$$Y(s^2+1) = s - 2 + \frac{1}{s^2}$$

$$Y = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s^2(s^2+1)}$$

$$\frac{s^2+1}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)}$$

$$Y = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$Y = \frac{s}{s^2+1} - 3\,\frac{1}{s^2+1} + \frac{1}{s^2}$$

$\mathcal{L}^{-1}$ $\quad y(t) = \underline{\cos t - 3\sin t + t}$

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$$y'' - 3y' + 2y = 4e^{2t} \quad \Big| \begin{array}{l} y(0) = -3 \\ y'(0) = 5 \end{array}$$

$\downarrow$

$$(s^2Y + 3s - 5) - 3(sY + 3) + 2Y = \frac{4}{s-2}$$

$$Y(s^2 - 3s + 2) + 3s - 5 - 9 = \frac{4}{s-2}$$

$$Y(s^2 - 3s + 2) = -3s + 14 + \frac{4}{s-2}$$

$$Y = \frac{-3s(s-2)}{(s-1)(s-2)^2} + \frac{14(s-2)}{(s-1)(s-2)^2} + \frac{4}{(s-1)(s-2)^2}$$

$$= \frac{-3s^2 + 6s + 14s - 28 + 4}{(s-1)(s-2)^2}$$

$$= \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

$$\frac{1}{s-\alpha} \quad e^{\alpha t}$$

$$Y = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\mathcal{L}^{-1}: \quad y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}$$