① Linearity $\mathcal{L}(\alpha f+\beta g)=\alpha\mathcal{L}(f)+\beta\mathcal{L}(g)$

$$\int_0^\infty \alpha e^{-st}f(t)\,dt+\int_0^\infty \beta e^{-st}g(t)\,dt =$$

$$\int_0^\infty e^{-st}\left(\alpha f(t)+\beta g(t)\right)dt$$

② $\mathcal{L}\left(e^{at}f(t)\right)=\int_0^\infty e^{at}e^{-st}f(t)\,dt$

$$=\int_0^\infty e^{-(s-a)t}f(t)\,dt \quad \text{Let } s'=s-a$$

frequency shift

$$=\int_0^\infty e^{-s't}f(t)\,dt=F(s')=F(s-a)$$

$\mathcal{L}(t f(t))$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$-\int_0^\infty e^{-st}\, t f(t)\,dt = (-1)\mathcal{L}(t f(t))$$

$$\mathcal{L}(t f(t)) = -F'(s)$$

$\underline{\mathcal{L}(t \sin \alpha t) \overset{?}{=}}$ Hint: $\mathcal{L}(\sin \alpha t) = \frac{\alpha}{s^2+\alpha^2}$

$$\frac{d}{ds}\left(\frac{\alpha}{s^2+\alpha^2}\right) = \frac{(s^2+\alpha^2)(0) - \alpha(2s)}{(s^2+\alpha^2)^2} =$$

$$+\frac{2\alpha s}{(s^2+\alpha^2)^2}$$

Base case established.

Ind. Hyp. $\mathcal{L}(t^k f(t)) = (-1)^k F^{(k)}(s)$

$$\mathcal{L}(t^K f(t)) = \int_0^\infty e^{-st} t^K f(t)\,dt = (-1)^K F^{(K)}(s)$$

$$\int_0^\infty \frac{\partial}{\partial s} e^{-st} t^K f(t)\,dt = (-1)^K F^{(K+1)}(s)$$

$$-\int_0^\infty e^{-st} t^{K+1} f(t)\,dt = (-1)^K F^{(K+1)}(s)$$

$$\mathcal{L}(t^{K+1} f(t)) = (-1)^{K+1} F^{(K+1)}(s)$$

If $\mathcal{L}(f(t)) = F(s) \Rightarrow \mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

Define $g(t) = \int_0^t f(u)\,du$, so...

$$g'(t) = f(t)\,;\ g(0) = 0$$

$$\mathcal{L}(g'(t)) = sG(s) - g(0) = sG(s)$$

$$F(s) = s\,G(s) = s\,\mathcal{L}\left\{\int_0^t f(u)\,du\right\}$$

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\xi)\,d\xi$$

Let $g(t) = \dfrac{f(t)}{t} \Rightarrow f(t) = t\,g(t)$

$$F(s) = (-1)\,G'(s)$$

$$\int F(\xi)\,d\xi = (-1)\,G(s) \Rightarrow G(s) = -\int_\infty^s F(\xi)\,d\xi$$

$$= \int_s^\infty F(\xi)\,d\xi$$

Choose integration constant so that

$$\lim_{s\to\infty} G(s) = 0$$

Periodic Extension:

$$\mathcal{L}(f(t)) = \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}} \quad T \text{ is period}$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt =$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt \ldots$$

Let $t = u+2T$

let $t = u+T$

$$= \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-s(u+T)} \underbrace{f(u+T)}_{f(u)}\,du$$

$$= \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-sT} \overbrace{e^{-su}}\, f(u)\,du + \int_0^T e^{-s(u+2T)} \ldots f(u+2T)\,du$$

⑥

$$\int_0^T e^{-su}f(u)\,du + e^{-sT}\int_0^T e^{-su}f(u)\,du + e^{-s(2T)}\int_0^T e^{-su}f(u)\,du + \cdots$$

$$\underbrace{\left(1+e^{-sT}+e^{-2sT}+e^{-3sT}+\cdots e^{-ksT}\cdots\right)}\int_0^T e^{-su}f(u)\,du$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

- $\mathcal{L}(f(t-a)) = \underline{e^{-as}}F(s)$ ..

$\downarrow$

$$\int_0^\infty \underline{e^{-st}}f(t-a)\,dt \quad \text{Let } u = t-a$$

$$= \int_{-a}^\infty e^{-s(u+a)}f(u)\,du = \int_{-a}^\infty e^{-sa}e^{-su}f(u)\,du$$

$$e^{-sa}\int_{-a}^\infty e^{-su}f(u)\,du = e^{-sa}\left[\int_{-a}^0 e^{-su}f(u)\,du + \int_0^\infty \text{ditto}\right]$$

Show $\int_0^\infty e^{-u^2}du = ? \ \frac{\sqrt{\pi}}{2}$

$I =$ so $I = \int_0^\infty e^{-v^2}dv$

$I^2 = \int_0^\infty\int_0^\infty e^{-u^2} \cdot e^{-v^2}\,du\,dv$

$\Downarrow$

* Thm $\iint f(x)g(y)\,dx\,dy = \int f(x)dx \cdot \int g(y)dy$

* $\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv = \left\{\begin{matrix} u = r\cos\theta \\ v = r\sin\theta \end{matrix}\right\}$

$\int_{0=\theta}^{\pi/2}\int_{r=0}^{\infty} \underline{e^{-r^2}}\, r\,dr\,d\theta$

$\int e^{-r^2} r\,dr \qquad u = r^2 \quad du = 2r\,dr$

$\int e^{-u} \frac{\cancel{r}}{2\cancel{r}} du = \frac{1}{2}\int e^{-u}du = \boxed{\frac{-e^{-u}}{2}}$

$$\int_0^{\pi/2} \left(\frac{1}{2}\right) d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$I = \int_0^{\infty} e^{-u^2} du = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$