Generating Functions:

What is a generating function?
"A zip file for a sequence"
Given sequence $a_0, a_1, \cdots a_i, \cdots$
we "encode" the sequence as coefficients
in a power series.

GF $a_0 \mu_0(x) + a_1 \mu_1(x) + \cdots + a_i \mu_i(x) + \cdots$

"indicator function"

Usual choice for $\mu_i(x)$ is x^i

This must happen $\sum_{n=0}^{\infty} a_n \mu_n(x) = \sum_{n=0}^{\infty} b_n \mu_n(x)$

implies $\underline{a_n = b_n}$
So indicator functions must satisfy this property

μ's : $\{1, 1+x, 1-x, 1+x^2, 1-x^2, \cdots\}$
Use this set to encode

Seq. 1 : 3, 2, 6, 0, 0, 0 →
Seq 2 : 1, 2, 6, 1, 1, 0, 0 →

Seq 1 : $3 \cdot 1 + 2 \cdot (1+x) + 6 \cdot (1-x) = -4x + 11$

Seq 2 : $1 \cdot 1 + 2 \cdot (1+x) + 6 \cdot (1-x) + 1(1+x^2) + 1(1-x^2)$

$= 11 - 4x$

Some typical indicator sets.

(i) x^n ordinary

(ii) $\frac{x^n}{n!}$. exponential

(iii) $\frac{x^n}{(n!)^2}$ doubly exponential

(iv) $\frac{x^n}{(n)!}$ — Eulerian where $n! : (1+q)(1+q+q^2) + \cdots (1+q+\cdots+q^{n-1})$

(v) $\frac{x^n}{q^{\binom{n}{2}} n!}$ chromatic

(vi) $\frac{x^n e^{-x}}{n!}$ poisson

(vii) $\frac{x^n}{1-x^n}$ lambert

Denote OGF (ordinary generating function) as $G(a_k; x)$ ← power series expansion of function defined by sequence $\{a_k\}_{k \in \mathbb{N}}$

Define "extraction operator"

$$[x^k]\, G(a_k; x) = a_k$$

$$a_0 = 1 \quad a_1 = n \quad a_2 = \binom{n}{2} \cdots a_n = \binom{n}{n}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

← ${}_nC_k$

Recall: ${}_nP_k = \dfrac{n!}{(n-k)!}$

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!} x^k = ?$$

$$\sum_{k=0}^{n} \left(\frac{n!}{(n-k)!}\right) \frac{x^k}{k!} \rightsquigarrow \sum_{k=0}^{n} \binom{n}{k} x^k =$$

$(1+x)^n$

Want general formula for the Fibonacci #'s:

$\{F_n\}$

Recurrence: $\sum F_{n+1} = \sum F_n + \sum F_{n-1}$ $\quad F_{-1} = 0,\ F_0 = 1$

Form OGF w/ $\sum_{n\geq 0} F_{n+1} x^n = \frac{1}{x} \sum_{n\geq 0} F_{n+1} x^{n+1}$

(i) $\frac{1}{x} \sum_{k\geq 1} F_k x^k$ (since we set $k = n+1$)

$= \frac{1}{x}\left[\sum_{k\geq 0} F_k x^k - F_0\right]$

(ii) $\sum_{n\geq 0} F_n x^n$

(iii) $\sum_{n\geq 0} F_{n-1} x^n = \sum_{n\geq 1} F_{n-1} x^n = x \sum_{n\geq 1} F_{n-1} x^{n-1}$

Let $\sum_{n\geq 0} F_n x^n = \phi(x)$

So... $\frac{1}{x}\left(\phi(x) - 1\right) = \phi(x) + x\phi(x)$

$$\phi(x) - 1 = x\phi(x) + x^2\phi(x)$$

$$-x^2\phi(x) - x\phi(x) + \phi(x) = 1$$

$$\phi(x)\left[1 - x - x^2\right] = 1$$

So... $\dfrac{1}{1-x-x^2} = \sum_{n\geq 0} F_n x^n$

Apply $F_n = [x^n]\left(\dfrac{1}{1-x-x^2}\right)$

$$(1-x-x^2) = (1-\alpha x)(1-\beta x)$$

$$1 - (\alpha+\beta)x + \alpha\beta x^2$$

$$-(\alpha+\beta) = -1 \Rightarrow \alpha+\beta = 1$$

$$\alpha\beta = -1$$

$$(1-\beta)\beta = -1 \quad \beta - \beta^2 = -1$$

$$\beta^2 - \beta - 1 = 0 \Rightarrow \beta = \frac{+1 \pm \sqrt{5}}{2}$$

Let $\beta = \dfrac{1+\sqrt{5}}{2}$; $\alpha = \dfrac{1-\sqrt{5}}{2}$

ϕ_1 $\qquad$ ϕ_2

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$$\frac{1}{1-\alpha x} = \sum_{n\geq 0}(\alpha x)^n \; ; \; \frac{1}{1-\beta x} = \sum_{n\geq 0}(\beta x)^n$$

So... $$\frac{1}{1-x-x^2} = \frac{1}{\sqrt{5}}\left[\sum_{n\geq 0}\alpha^{n+1}x^n - \sum_{n\geq 0}\beta^{n+1}x^n\right]$$

Then $$[x^n]\left(\frac{1}{1-x-x^2}\right) = \frac{1}{\sqrt{5}}\left(\alpha^{n+1}-\beta^{n+1}\right)$$

or $$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$ ∎

$$(1+x)^r = 1 + \sum_{k=1}^{\infty}\frac{r(r-1)\cdots(r-k+1)}{k!}\cdot x^k$$

$E = mc^2$ $\quad m(v) = m_0 \cdot \frac{1}{\sqrt{1-v^2/c^2}}$

Let $\beta = \frac{v}{c}$

$$\downarrow$$

$$m_0\underline{(1-\beta^2)^{-1/2}}$$

$x = -\beta^2 \quad r = -\frac{1}{2}$

$$m(v) = 1 + \frac{(-\frac{1}{2})(-\beta^2)}{1!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(\beta^4)}{2!}$$

$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\beta^6)}{3!} + O(\beta^8)$$

$$= 1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 + \frac{15}{48}\beta^6 + O(\beta^8)$$

$E = m_0 c^2$

$$E = m_0c^2\left(1 + \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{15}{48}\frac{v^6}{c^6} + O\left(\left(\frac{v}{c}\right)^8\right)\right)$$

$$E = \underbrace{m_0c^2}_{\text{rest energy}} + \underbrace{\left[\underbrace{\frac{m_0v^2}{2}}_{\text{classical kinetic}} + \frac{3}{8}m_0\frac{v^4}{c^2} + \cdots\right]}_{\text{relativistic KE}}$$