Recurrence Relations

$$(*)\ c_0 a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n)$$

$a_i \in \mathbb{Z} \quad c_0 \in \mathbb{Z}$

$a_1, a_2, a_3, \cdots, a_i, \cdots$

1. $a_n = 2a_{n-1}$; $a_0 = 1$

Assume $a_n = cr^n$

$cr^n = 2cr^{n-1}$ or $r^n - 2r^{n-1} = 0$

divide by r^{n-1} : $r - 2 = 0 \Rightarrow r = 2$

$a_n = c2^n$

fit initial condition

$a_0 = 1 = c \cdot 2^0 = c$ so $c = 1$

$$\boxed{a_n = 2^n}$$

F_n = # bunnies end month n

F_{n-1} = # bunnies end month $n-1$

F_{n-2} = # bunnies **produced** in month n

$$F_n = F_{n-1} + F_{n-2}$$

1, 1, 2, 3, 5, 8, ...

$$F_n - F_{n-1} - F_{n-2} = 0$$

$F_n = Cr^n$

$$\cancel{C}r^n - \cancel{C}r^{n-1} - \cancel{C}r^{n-2} = 0$$

$\div r^{n-2} \Rightarrow \quad r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$$F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_0 = 0 = C_1 + C_2 \Rightarrow \underline{C_1 = -C_2}$$

$$F_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) - C_1\left(\frac{1-\sqrt{5}}{2}\right)$$

C_1

$$F_1 = 1 = \overset{\varphi_0}{C_1\left(\frac{1+\sqrt{5}}{2}\right)} - \overset{\varphi_1}{C_1\left(\frac{1-\sqrt{5}}{2}\right)}$$

$$1 = C_1\left[\sqrt{5}\right] \Rightarrow C_1 = \frac{\sqrt{5}}{5}$$

Final: $F_n = \dfrac{\sqrt{5}}{5}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$ ✓

Binet formula $\quad 1.618\ldots \quad 0.618\ldots$

$F_{100} = ?\ 3.54 \times 10^{20}$.

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$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1.618\sim$$

Lucas sequence $L_n = L_{n-1} + L_{n-2}$

$L_1 = 1 \quad L_2 = 3$

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solve $\underline{a_n + 3a_{n-1}} = \underline{n+5}$ $[a_0 = 1]$

$$a_n + 3a_{n-1} = 0$$

$$a_n = Cr^n \quad \cancel{C}r^n + 3\cancel{C}r^{n-1} = 0$$

$$r = -3$$

$$a_n^h = C(-3)^n$$

$$a_0^h = 1 = C(-3)^0 \Rightarrow C = 1$$

$$a_n^h = (1)(-3)^n$$

$$a_n^p = An + B$$

$$\underbrace{(An+B)}_{a_n} + 3\underbrace{(A(n-1)+B)}_{a_{n-1}} = n + 5$$

(i) $4A\cancel{n} = \cancel{n} \Rightarrow A = \frac{1}{4}$

(ii) $B - 3A + 3B = 5$

$$4B = 5\tfrac{3}{4} \Rightarrow B = \frac{23}{16}$$

$$a_n = (-3)^n + \frac{n}{4} + \frac{23}{16}$$

Solve: $a_n + 4a_{n-1} + 4a_{n-2} = 0$

$\underline{a_0 = 1 = a_1}$

$$r^n + 4r^{n-1} + 4r^{n-2} = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \quad r = -2 \ (\times 2)$$

$$a_n = B_1(-2)^n + nB_2(-2)^n$$

$$a_n = (B_1 + nB_2)(-2)^n$$

$$1 = B_1(-2)^0 \Rightarrow B_1 = 1$$

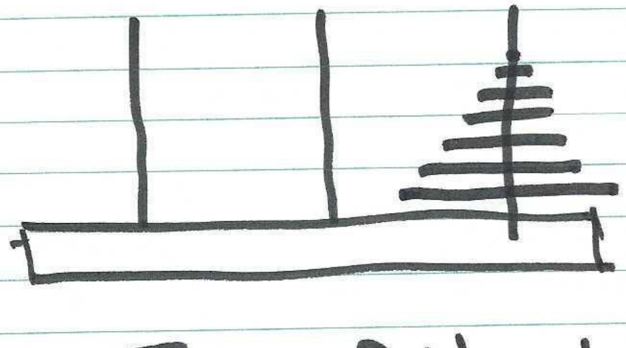
$$1 = (B_1 + B_2)(-2)$$

$$1 = (B_2 + 1)(-2) \Rightarrow$$

$$B_2 + 1 = -\frac{1}{2} \Rightarrow B_2 = -\frac{3}{2}$$

$$a_n = (-2)^n - \frac{3n}{2}(-2)^n$$

$$= \left(1 - \frac{3n}{2}\right)(-2)^n$$



Tower of Hanoi