S2W84

9-3

①

Test #1 will be thru Ch. 4 only ←

Loose ends from Ch. 3:

① The center of a group is a subgroup

$Z(G) := \{g \in G : gh = hg, \forall h \in G\}$

Pf: Want to show $Z(G) \leq G$

Use 1-step test:

Suppose $a, b \in Z(G)$ and we form ab^{-1}. Question is, is $ab^{-1} \in Z(G)$

This would $\forall h \in G$, $hab^{-1} = ab^{-1}h$.

$\underline{ha}b^{-1} = a\underline{hb^{-1}} = \underline{ab^{-1}h}$. So $Z(G)$ is a subgroup. ∎

② The centralizer of an element $g \in G$ is a subgroup of G.

(2)

$C(g) = \{h \in G : gh = hg\}$

Claim: $C(g) \leq G$

Use 1-step test.

Pick $a, b \in C(g)$, form $ab^{-1} \overset{?}{\in} C(g)$

For this to be true, $g\underline{ab}^{-1} = \underline{ab}^{-1}g$

$gab^{-1} = agb^{-1} = ab^{-1}g$ ✓ So $C(g) \leq G$ ■

Let G be a group. Then :

$$Z(G) = \bigcap_{g \in G} C(g)$$

Cyclic groups:

We say the element a "generates" a group G (and write this as $\langle a \rangle = G$) if taking all powers of a produces all of G.

If $g, g^2, \cdots g^{n-1}, g^n = G$

What is g^{n+1}?

$\langle e \rangle = \{e\} = G$

$\mathbb{Z}_n$ groups are cyclic.

If n is a prime number, every non-id generates $\mathbb{Z}_n$.

$\mathbb{Z}_5 : \{0, 1, 2, 3, 4\}$ (addition mod 5)

id

$1+1+1+1+1 = 5 = 0$

2
$2+2 = 4$
$2+2+2 = 1$
$2+2+2+2 = 3$
$2+2+2+2+2 = 10 \bmod 5 = 0$

$\mathbb{Z}_4 : \{0, 1, 2, 3\}$ add'n mod 4

1
$1+1 = 2$
$1+1+1 = 3$
$1+1+1+1 = 4 \bmod 4 = 0$

2
$2+2 = 4 \equiv 0$

Theorem: If p is prime, then $\mathbb{Z}_p$ is cyclic with $p-1$ generators

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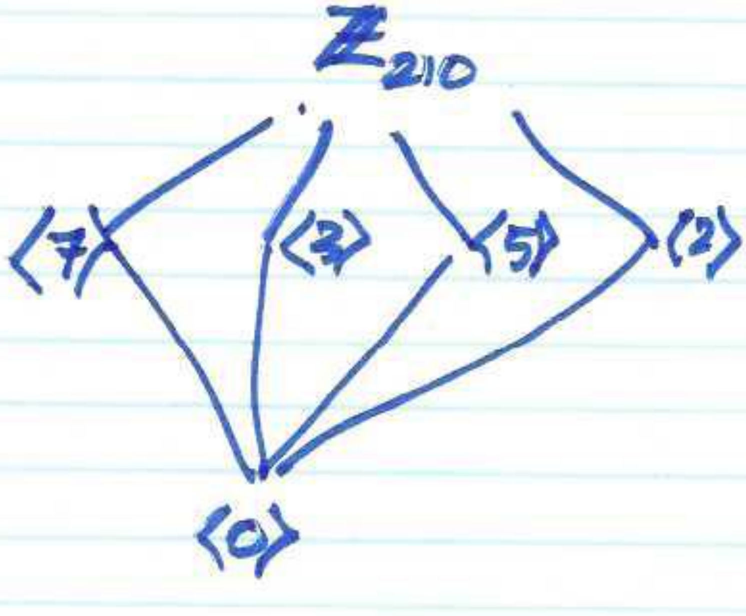
Fact: If $\langle g \rangle$ has order n, there is a subgroup for every divisor of n.

So if $\underline{k | n}$, then $\exists H \leq G$ with order $\frac{n}{k}$

k divides n

$\mathbb{Z}_{210}$ $210 = 7 \cdot 5 \cdot 3 \cdot 2$



⑥

The Euler ϕ-function (also called the totient function) is defined as follows.

Pick 5 $\underline{\mathbb{Z}_5 = \{0,1,2,3,4\}}$

$\phi(5) = 4$ means we should have 4 generators (i.e. rel. prime to 5)

1
$1+1=2$
$1+1+1=3$
$1+1+1+1=4$
$\overline{}$ $5 \equiv 0$

$\mathbb{Z}_{10}$ - how many generators

$\phi(10) = \underline{4}$

$U(50)$ $\qquad$ $50 = 2 \cdot 5^2$

Property of ϕ function is :

$\rightarrow \phi(2 \cdot 5^2) = \phi(2) \cdot \phi(5^2) = \underline{\underline{(1) \cdot (20)}}$

* so ϕ is multiplicative.
* Also $\phi(p^n) = p^n - p^{n-1}$

Consider the element 3 in $U(50)$

$|U(50)| = 20 = \phi(2 \cdot 5^2)$

Find all generators of $\mathbb{Z}_6$, $\mathbb{Z}_8$, & $\mathbb{Z}_{20}$

$\uparrow$

$\langle 2 \rangle = \{0, 2, 4, 6\}$ $\qquad$ $\langle 6 \rangle = \{0, 6, 4, 2\}$

$\langle 4 \rangle = \{0, 4\}$

$0 \mid \underbrace{1 \quad 3 \quad 5 \quad 7}_{\text{gen}} \qquad \underbrace{2, 4, 6}_{\text{not gen}}$

8

$\mathbb{Z}_{20}$ $\qquad 20 = 2^2 \cdot 5$

$$\phi(20) = \underbrace{(2^2-2) \cdot \phi(5)}_{8}$$

$$\begin{cases} ① \ \phi(pq) = \phi(p) \cdot \phi(q) \\ ② \ \phi(p^n) = p^n - p^{n-1}. \end{cases}$$

$\mathbb{Z}_{24}$ $\qquad 24 = 3 \cdot 2^3$

$$\phi(24) = \underset{2}{\phi(3)} \cdot 4 = 8$$

Show $\langle a \rangle = \langle a^{-1} \rangle$

$a, a^2, a^3, \ldots$

$a^{-1}, a^{-2}, a^{-3} \ldots$ same