H86VH

Isomorphism

Morphism means "shape" Iso means "same"

Iso m/ : a bijective mapping that is "operation preserving"

Given group G & another group $\bar{G}$

To establish an isomorphism between G & $\bar{G}$,

1) Find a candidate mapping $\phi : G \to \bar{G}$

2) Show that ϕ is 1:1 (injective)

3) Show that ϕ is surjective / onto

4) Establish "operation preservation"

Suppose $\langle G, * \rangle$ & $\langle \bar{G}, \# \rangle$

O.P. condition $g_1, g_2 \in G$ then

$$(*) \quad \phi(g_1 * g_2) = \phi(g_1) \# \phi(g_2)$$

$G = \langle \mathbb{R}, + \rangle \quad \bar{G} = \langle \mathbb{R}_+, \cdot \rangle$

$\overset{?}{} \ \phi(r_1 + r_1) = \phi(r_1) \cdot \phi(r_2)$

$\phi(r) = e^r \qquad\qquad e^{r_1+r_2} = e^{r_1} \cdot e^{r_2}$

bases: $b > 0$
$\neq 1$

Checklist:

1) mapping has proper domain/range

1:1 2) Say $e^r = e^s$, need to show $r = s$

$\ln(e^r) = r \quad \ln(e^s) = s \quad r = s$!

onto 3) Given $e^r \in \bar{G}$, inverse is $r \in G$

O.P. 4) $\phi(r_1 + r_2) = \phi(r_1) \cdot \phi(r_2)$

$e^{r_1+r_2} = e^{r_1} \cdot e^{r_2}$

Given $G, \bar{G}$ w/ $\phi : G \to \bar{G}$ an iso^{m}

$$\phi(e_G) = e_{\bar{G}}$$

$$\phi(e_G * e_G) = \phi(e_G) \# \phi(e_G) \quad \Big| \ \phi \text{ is iso}^{m}$$

$$\downarrow$$

in $\bar{G}$ $\quad \underline{\phi(e_G)} = \underline{\phi(e_G)} \# \underline{\underline{\phi(e_G)}}$

Th^{m} identities are preserved under iso^{m}

Th^{m} 2) If G is commutative, $\bar{G}$ is as well and v.v.

G commutes $\left\{ \begin{array}{l} \phi(a * b) = \phi(a) \# \phi(b) \\ \quad \shortparallel \\ \phi(b * a) = \phi(b) \# \phi(a) \end{array} \right.$ $\bar{G}$ commutes too

Th^{m} If ϕ is an iso^{m}, so is $\phi^{-1} : \bar{G} \to G$

Thm If G is cyclic, so is $\bar{G}$

If G is cyclic, then $\exists\, a \in G$ s.t.

$\underline{g = a^n}$ $\underline{a \text{ is generator}}$

Suppose

$$\phi(g) = \phi(\overbrace{a \cdot a \cdot a \cdots a}^{n}) =$$

$$\underbrace{\phi(a)\,\phi(a)\,\phi(a) \cdots \phi(a)}_{n \text{ factors}}$$

$\phi(g) \in \bar{G}$

If a is generator for G, $\underline{\phi(a)}$ generates $\bar{G}$

Thm Given g in G, if $|g| = n$, then

$\phi(g) \in \bar{G}$ also has order n.

$\underbrace{g \cdot g \cdots g}_{n} = e_G$ | $\phi(g^n) = \underline{\phi(g)\,\phi(g) \cdots \phi(g)}$

$\phi(e_G) = {}'' e_{\bar{G}}$

If $\phi : G \to \bar{G}$ is an isomorphism, and $|G| < \infty$, then each group has the same number of elements of a given order.

Th^m $\phi\big(Z(G)\big) = Z(\bar{G})$

Given $a \in Z(G)$ then $\forall g \in G$, $\underline{ag = ga}$

Want to show $\phi(a) \in Z(\bar{G})$

For every $g \in G$,
$$\begin{array}{ccl} \phi(ag) & = & \phi(a)\phi(g) \\ \| & & \\ \phi(ga) & = & \phi(g)\phi(a) \end{array}$$

∎

Th^m If $K \leq G$, then $\phi(K) = \{\phi(k) : k \in K\}$ is a subgroup of $\bar{G}$

Short version: If $K \leq G$, $\phi(K) \leq \bar{G}$

Suppose $a, b \in K$, then $\underline{ab^{-1} \in K}$

Consider $\underline{\phi(ab^{-1})} = \phi(a)[\phi(b)]^{-1} \in \phi(K)$

Why is $\phi(g^{-1}) = [\phi(g)]^{-1}$

$$\phi(g^{-1}g) = \phi(e_G) = e_{\bar{G}}$$

$\downarrow$ O.p.

$$\phi(g^{-1}) \cdot \phi(g) = e_{\bar{G}}$$

$\downarrow$

$$\underline{\phi(g^{-1}) = [\phi(g)]^{-1}}$$

Automorphism : $\phi : G \to G$

that is an isomorphism