 8-25

Axioms for Group

1) Given set G and binary operation $*$

Associative 2) $a,b,c \in G$ then $a*(b*c) = (a*b)*c$

Identity 3) $\exists e \in G \overset{s.t.}{\ni} \forall g \in G,\ e*g = g*e = g$

Inverse 4) $\forall g \in G\ \exists g^{-1} \ni g*g^{-1} = g^{-1}*g = e$

Commutative 5) $\forall a,b \in G,\ a*b = b*a$

$\mathbb{N}$ whole numbers $\mathbb{N}_0$ (with zero)

$\mathbb{Z}$ integers is a group under $+$ (not $\cdot$)

$\mathbb{Q}$ rationals group under $+$, and $\setminus\{0\}$ under $\cdot$

$\mathbb{R}$ reals " "

$\mathbb{C}$ complex numbers " "

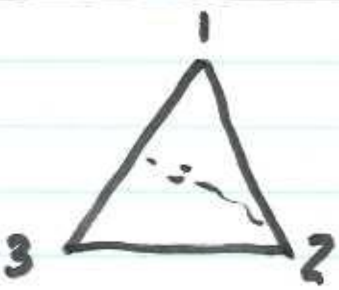
$$(a+bi)^{-1} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} \overset{?}{=} x+yi$$

Differential Operators – not group under composition

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots$

Consider: $\frac{\partial}{\partial x}\left(e^{x^2 y}\right)$ $\quad \int e^{x^2 y}\cdot 2x\,dx \longrightarrow$ OK

$\int e^{x^2}\,dx$ DNE



$$D_6 \left\{ \begin{array}{ll} F_1 & R_{120} \\ F_2 & R_{240} \\ F_3 & R_0 \end{array} \right.$$

$\frac{\text{3-gon}}{\text{n-gon}}$

Checklist:

1) Associative ✓
by functional composition

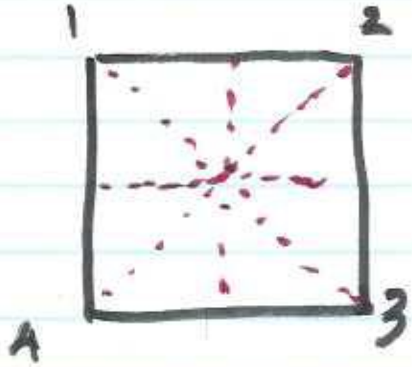
2) R_0 is identity

3) F_1, F_2, F_3, R_0 are own inverses

$R_{120}^{-1} = R_{240}$ and v.v.

Symmetry operations on n-gons are called "dihedral" groups D_{2n}

Note: the number of elements in a group is called its "order." $|D_{2n}| = 2n$



D_8

$$\left\{ \begin{matrix} F_V & F_H & F_D & F_A \\ R_0 & R_{90} & R_{180} & R_{270} \end{matrix} \right\}$$

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right

$x \circ y$

left

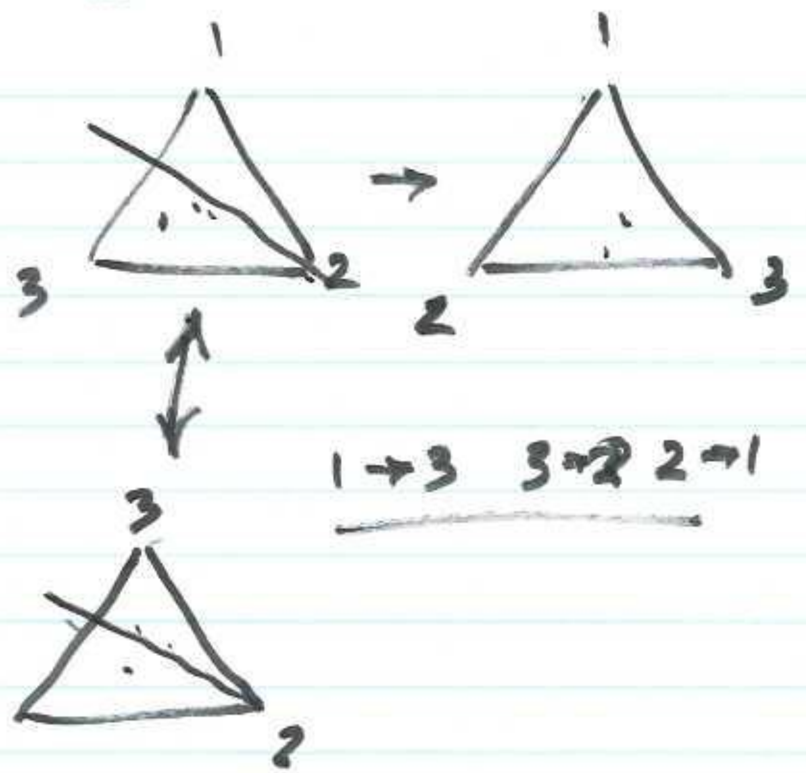
	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_0	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	R_0	F_3	...	
R_{240}	R_{240}	R_0	R_{120}			
F_1	F_1					
F_2	F_2					
F_3	F_3					

Arthur Cayley

Cayley Table

④

$R_{120} \circ F_1$



$1 \to 3 \quad 3 \to 2 \quad 2 \to 1$

$R_{120} \circ F_1 = F_2$

$R_0 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \leftarrow \begin{pmatrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a' & b' & c' \end{pmatrix}$

$R_{120} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$R_{240} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$F_1 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$F_2 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$F_3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

	1	2	3
F_1	1	3	2
R_{120}	2	1	3

$R_{120} \circ F_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

F_3

$F_1 \circ R_{120} = ?$

$$\begin{array}{cccc} & 1 & 2 & 3 \\ R_{120} & 2 & 3 & 1 \\ F_1 & 3 & 2 & 1 \end{array} \qquad \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} = F_2$$

Symmetric Group S_n

So 3 objects generate S_3

Note $D_6 = S_3$ & $D_{2n} = S_n$ only $n=3$

$D_8 \neq S_4$ $|S_4| = 4! = 24$

$\{e, a, b, c\}$

C_4

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$$\boxed{\begin{aligned} a &= a \\ a^2 &= b \\ a^3 &= c \\ a^4 &= e \end{aligned}}$$

<u>cyclic</u>

Klein group

V_4

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\underline{e^2 = a^2 = b^2 = c^2 = e}$

V ~ <u>vierergruppe</u> 4-group

Table 10.1 Modular Arithmetic Operations
x,y,u,v,k,r∈ , k≠0

1) Addition	if x≡y(modm) and u≡v(modm) then x+u≡y+v(modm)
2) Subtraction	if x≡y(modm) and u≡v(modm) then x-u≡y-v(modm)
3) Multiplication	if x≡y(modm) and u≡v(modm) then xu≡yv(modm)
4) Division	if x≡y(modm) and x=ku, y=kv then (x/k)≡(y/k)(modm) if gcd(k,m)=1
5) Powers	if x≡y(modm) then x^{r}≡y^{r}(modm)

$x \equiv y \pmod m \qquad x^r \equiv y^r \pmod m$

$7^4 \equiv 1 \pmod{100}$ so... $7^{4k} \equiv 1 \pmod{100}$

We say $x \equiv y \pmod m \Leftrightarrow x - y = km \ \exists k \in \mathbb{Z}$