YUWXB

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Th^{m} If $\phi: G \to \bar{G}$ is a homomorphism and $\ker\phi \leq G$. Show $\ker\phi \triangleleft G$.

Pf: Pick $x \in \ker\phi$. Want to show that $\forall g \in G$, $gxg^{-1} \in \ker\phi$. Test: is $\phi(gxg^{-1}) = e_{\bar{G}}$

$\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g^{-1}) =$

$\phi(g)\phi(x)[\phi(g)]^{-1} = \phi(g)[\phi(g)]^{-1} = e_{\bar{G}}$

$\uparrow$ $e_{\bar{G}}$

Conclude $gxg^{-1} \in \ker\phi$, i.e. $\ker\phi \triangleleft G$.

Th^{m} Given $N \triangleleft G$, then $\exists\, \phi$ s.t. N is $\ker\phi$.

Pf: Def^{n} the "natural map" is the mapping $G \to G/N$ ← normal subgroup

So $\phi: G \to G/N$

FIRST ISOMORPHISM THEOREM
Let $\phi : G \to H$ be a group homomorphism. Then $G/\ker\phi \cong \mathrm{Im}\,\phi$

BACKGROUND:

Camille Jordan was apparently aware of this theorem in a limited context by 1870, but the first time anyone wrote it down in the form we recognize today was in a 1927 paper by Emmy Noether. Hölder's Theorem, which says that if a subgroup is normal its cosets have a natural operation under which they form a group themselves...the factor group...was not available until 1888. The First Isomorphism Theorem is sometimes considered to be part of the Fundamental Homomorphism Theorem for groups, which subsumes the first, second, and third isomorphism theorems. That is convenient, since there does not seem to be any agreement after eighty years on which of the second two parts is to be called the Second Isomorphism Theorem, and which is the Third Isomorphism Theorem. You will see them labelled either way, depending on the textbook. As with many "first, second, third" theorems, the first one is the most significant and the others depend on it. The Fourth Isomorphism Theorem, also known as Zassenhaus' Lemma or the Butterfly Theorem from the Hasse diagram of its constituent groups, is not part of this bundle.

The First Isomorphism Theorem creates an isomorphism where there wasn't one by restricting the codomain to the range of a given homomorphism, thus ensuring surjectivity, and "modding out" the kernel of that homomorphism to obtain injectivity. Fortunately, the image of the domain group under a homomorphism is a subgroup of the codomain group, and the kernel of a homomorphism is a normal subgroup of the domain group, so the factor group construction makes sense. The punchline of the theorem is that the image group and factor group are essentially the same.

KEY DEFINITIONS:

1) A *group homomorphism* is an operation preserving mapping between two groups, specifically if G and H are groups and $\phi : G \to H$, then ϕ is a homomorphism iff for all $x, y \in G$ it is true that $\phi(xy) = \phi(x)\phi(y)$. Another way to state the operation preserving property is to say that the group operations and the mapping commute.

2) The *kernel* of a group homomorphism is the preimage or pullback of the identity in the codomain group. Arbitrary maps between groups don't necessarily have kernels...only homomorphisms, so that the operation preserving feature of the homomorphism ensures that the kernel is a subgroup. If $\phi : G \to H$ is a homomorphism, then $\ker\phi = \{g \in G : \phi(g) = e_H\}$, where e_H is the identity in H.

3) A group *isomorphism* is a bijective group homomorphism. Group isomorphisms give all of the algebraic properties of the group on one side of the mapping to the group on the other side. Be careful to realize that properties like order and boundedness are not algebraic and do not need to be preserved by isomorphisms.

4) A *coset* is formed by taking an arbitrary element of a group, and operating with that element on every element in a chosen subgroup. If the arbitrary element operates from the left, we get a left coset of the subgroup, and if it operates from the right we get a right coset. It turns out that cosets of a given subgroup are mutually disjoint and all the same size.

5) A subgroup H is *normal* in the group G (written $H \triangleleft G$) if for any $g \in G$ and $h \in H$, the product $ghg^{-1} \in H$. Intuitively speaking, *normality* measures how mixed up the elements of a subgroup can become when they are operated first on the left by an arbitrary element of the group

and then on the right by its inverse. If you get back to the subgroup every time, then the subgroup is normal (an older word is "invariant", which explicitly conveys this stability property). There is no guarantee that you will, so this is a special situation and it has many useful consequences. Another intuitive viewpoint is that the elements of H "almost commute" with the elements of G, that is $gh = h'g$.

KEY FACTS:

1) Hölder's Theorem : If $H \lhd G$, then $G/H = \{gH : g \in G\}$ is a group where the operation is given by $(xH)(yH) = xyH$.

2) The kernel of a homomorphism is a normal subgroup of the domain.

PROOF STRATEGY:

This proof is constructive...we build an isomorphism out of the raw material available. The most important thing we have going for us is the operation preserving map...the homomorphism...which might be turned into an isomorphism if the defects in its bijectivity can be repaired. Repairing surjectivity is easy, just restrict the codomain to the image. Repairing injectivity is less obvious, but we collapse all elements in the domain that get mapped to a common element in the range by the expedient of forming cosets of the kernel. Cosets of the kernel can't distinguish such elements, and as far as the cosets are concerned the mapping is injective.

PROOF:

Since $\ker\phi \lhd G$, it makes sense to consider the factor group $G/\ker\phi$. Since we are looking for a natural association between cosets of this kernel and elements of H, I propose that we investigate the mapping $\Psi : G/\ker\phi \to H$, given by $x\ker\phi \to \phi(x)$. This is surely the most obvious guess. In words, this mapping sends a coset to the image of its representative. Now there might be a problem here if we picked two different representatives for the same coset and they wind up being mapped to distinct range elements. Then we wouldn't even have a function. But our fears are unfounded, since if $x\ker\phi = y\ker\phi$, then $\phi(x) = \phi(x)\phi(\ker\phi) = \phi(x\ker\phi) = \phi(y\ker\phi) = \phi(y)\phi(\ker\phi) = \phi(y)$. This chain of equalities is justified by the operation preserving property of the mapping ϕ and the fact that $\phi(\ker\phi) = e_H$. We can now assert that our new mapping Ψ is at least well-defined....i.e. is a function!

Is Ψ injective? If $\phi(x) = \phi(y)$, then $\phi(x)[\phi(y)]^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1}) = e_H$, but this means $xy^{-1} \in \ker\phi$. In turn it must be that $x \in y\ker\phi$, hence $x\ker\phi \subset y\ker\phi$. Reversing the roles of x and y, we have $y\ker\phi \subset x\ker\phi$, so $y\ker\phi = x\ker\phi$, and we have established that the mapping Ψ is injective.

Now any injective map is bijective onto its range, and clearly the range of Ψ is precisely the range of ϕ. So trimming the codomain from H down to $\mathrm{Im}\,\phi$ makes Ψ bijective.

Finally, is Ψ operation preserving? In other words is $\Psi(xHyH) = \Psi(xH)\Psi(yH)$. Well, $\Psi(xHyH) = \Psi(xyH) = \phi(xy) = \phi(x)\phi(y) = \Psi(xH)\Psi(yH)$, so yes.

Now we have established that Ψ is a well-defined operation preserving bijection, which is sufficient to declare Ψ to be an isomorphism, so $G/\ker\phi \overset{\Psi}{\cong} \mathrm{Im}\,\phi$.

Recall that the homomorphic image of a group is always a group as well, but it may be far from isomorphic to the original...in fact it could collapse a really big group into the identity. This theorem allows us to manufacture an isomorphism where there was none.

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SECOND ISOMORPHISM THEOREM

Let G be a group where $H \triangleleft G$, $N \triangleleft G$, and $N \subset H$. Then $\frac{G/N}{H/N} \cong G/H$.

BACKGROUND:

See the comments for the First Isomorphism Theorem. It looks superficially like we are "cancelling" the subgroup N by some arithmetic operation, whatever that would mean in this context (i.e. nothing!), although it is helpful as a mnemonic. In reality, we are forming three factor groups and observing that they stand in this relationship to one another by virtue of the First Isomorphism Theorem.

KEY DEFINITIONS:

1) A *group homomorphism* is an operation preserving mapping between two groups, specifically if G and H are groups and $\phi : G \to H$, then ϕ is a homomorphism iff for all $x, y \in G$ it is true that $\phi(xy) = \phi(x)\phi(y)$. Another way to state the operation preserving property is to say that the group operations and the mapping commute.

2) The *kernel* of a group homomorphism is the preimage or pullback of the identity in the codomain group. Arbitrary maps between groups don't necessarily have kernels...only homomorphisms, so that the operation preserving feature of the homomorphism ensures that the kernel is a subgroup. If $\phi : G \to H$ is a homomorphism, then $\ker\phi = \{g \in G : \phi(g) = e_H\}$, where e_H is the identity in H.

3) A group *isomorphism* is a bijective group homomorphism. Group isomorphisms give all of the algebraic properties of the group on one side of the mapping to the group on the other side. Be careful to realize that properties like order and boundedness are not algebraic and do not need to be preserved by isomorphisms.

4) A *coset* is formed by taking an arbitrary element of a group, and operating with that element on every element in a chosen subgroup. If the arbitrary element operates from the left, we get a left coset of the subgroup, and if it operates from the right we get a right coset. It turns out that cosets of a given subgroup are mutually disjoint and all the same size.

5) A subgroup H is *normal* in the group G (written $H \triangleleft G$) if for any $g \in G$ and $h \in H$, the product $ghg^{-1} \in H$. Intuitively speaking, *normality* measures how mixed up the elements of a subgroup can become when they are operated first on the left by an arbitrary element of the group and then on the right by its inverse. If you get back to the subgroup every time, then the subgroup is normal (an older word is "invariant", which explicitly conveys this stability property). There is no guarantee that you will, so this is a special situation and it has many useful consequences. Another intuitive viewpoint is that the elements of H "almost commute" with the elements of G, that is $gh = h'g$.

KEY FACTS:

1) Hölder's Theorem : If $H \triangleleft G$, then $G/H = \{gH : g \in G\}$ is a group where the operation is given by $(xH)(yH) = xyH$. This is the *factor group* of G by H.

2) The kernel of a homomorphism is always a normal subgroup of the domain.

3) First Isomorphism Theorem: If $\phi : G \to H$ is a group homomorphism. Then $G/\ker\phi \cong \mathrm{Im}\,\phi$

PROOF STRATEGY:

The proof is a straightforward application of the First Isomorphism Theorem. Once we get the pieces defined properly, it is immediate. We will invent a homomorphism from G/N to G/H that has H/N as its kernel.

PROOF:

First, notice that $N \subset H$ and $H \triangleleft G$ imply $N \triangleleft H$. Why? We are given $N \triangleleft G$, which means $gng^{-1} \in N$. But every $h \in H$ is also a g, so $hnh^{-1} \in N$ as well. So it makes sense to form the factor group H/N. Now the factor groups G/H and G/N are available immediately from Hölder's Theorem, and the only thing left to do is explain why the three of them may be configured as we claim. One loose end at the moment is the expression $\frac{G/N}{H/N}$. In order for it to make sense, we have to establish that $H/N \triangleleft G/N$. Let's try to hit two birds with one stone. If we can dream up a homomorphism from G/N that has H/N as its kernel and G/H as its image, not only will the "factor group of factor groups" $\frac{G/N}{H/N}$ be well-defined, but the *First* Homomorphism Theorem will immediately give us the fact that it is isomorphic to its image G/H, which is the entire claim of the Second Isomorphism Theorem.

OK... here is a candidate mapping... $\phi : G/N \to G/H$ which sends the coset gN to the coset gH. The mapping just strips out the coset representative g, tosses away N, then applies g to the subgroup H to form a bigger coset gH. We better establish that this mapping is actually a homomorphism before we go any further. To do so, we need to verify that $\phi(g_1Ng_2N) = \phi(g_1N)\phi(g_2N)$. Well, on the left hand side normality allows us to write $\phi(g_1Ng_2N) = \phi(g_1[g_2N]N) = \phi(g_1g_2N) = g_1g_2H$. And on the right hand side $\phi(g_1N)\phi(g_2N) = (g_1H)(g_2H) = g_1[Hg_2]H = g_1[g_2H]h = g_1g_2H$. So ϕ is a homomorphism and we are in business.

What is the kernel of ϕ? It is whatever gets mapped to the identity of G/H, which is, of course, just H. Consider what ϕ does to the coset $hN \in G/N$. It sends it to $hH = H$! So all the elements (and only those) of H/N, namely the cosets of the form hN, are mapped onto H. This allows us to conclude immediately that $H/N \triangleleft G/N$ by the second key fact above. Now $\frac{G/N}{H/N}$ is a legitimate construction with $\ker\phi = H/N$.

What is the image of ϕ? Certainly any coset gH has a pullback gN, so $\operatorname{Im}\phi = G/H$. The First Isomorphism Theorem gives us that $G/N/\ker\phi \cong \operatorname{Im}\phi$, or $\frac{G/N}{H/N} \cong G/H$.

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THIRD ISOMORPHISM THEOREM

Let G be a group where $H \leq G$ and $N \triangleleft G$. Then $\frac{H}{H \cap N} \cong \frac{HN}{N}$



BACKGROUND:

See the comments for the First Isomorphism Theorem.

$HN = \{hn : h \in H, n \in N\}$

KEY DEFINITIONS:

1) A *group homomorphism* is an operation preserving mapping between two groups, specifically if G and H are groups and $\phi : G \to H$, then ϕ is a homomorphism iff for all $x,y \in G$ it is true that $\phi(xy) = \phi(x)\phi(y)$. Another way to state the operation preserving property is to say that the group operations and the mapping commute.

2) The *kernel* of a group homomorphism is the preimage or pullback of the identity in the codomain group. Arbitrary maps between groups don't necessarily have kernels...only homomorphisms, so that the operation preserving feature of the homomorphism ensures that the kernel is a subgroup. If $\phi : G \to H$ is a homomorphism, then $\ker\phi = \{g \in G : \phi(g) = e_H\}$, where e_H is the identity in H.

3) A group *isomorphism* is a bijective group homomorphism. Group isomorphisms give all of the algebraic properties of the group on one side of the mapping to the group on the other side. Be careful to realize that properties like order and boundedness are not algebraic and do not need to be preserved by isomorphisms.

4) A *coset* is formed by taking an arbitrary element of a group, and operating with that element on every element in a chosen subgroup. If the arbitrary element operates from the left, we get a left coset of the subgroup, and if it operates from the right we get a right coset. It turns out that cosets of a given subgroup are mutually disjoint and all the same size.

5) A subgroup H is *normal* in the group G (written $H \triangleleft G$) if for any $g \in G$ and $h \in H$, the product $ghg^{-1} \in H$. Intuitively speaking, *normality* measures how mixed up the elements of a subgroup can become when they are operated first on the left by an arbitrary element of the group and then on the right by its inverse. If you get back to the subgroup every time, then the subgroup is normal (an older word is "invariant", which explicitly conveys this stability property). There is no guarantee that you will, so this is a special situation and it has many useful consequences. Another intuitive viewpoint is that the elements of H "almost commute" with the elements of G, that is $gh = h'g$.

KEY FACTS:

1) Hölder's Theorem : If $H \triangleleft G$, then $G/H = \{gH : g \in G\}$ is a group where the operation is given by $(xH)(yH) = xyH$. This is the *factor group* of G by H.

2) The kernel of a homomorphism is always a normal subgroup of the domain.

3) The intersection of two subgroups of a given group is also a subgroup.

4) If $H,N \leq G$ and $N \triangleleft G$, then $HN = \{hn : h \in H, n \in N\} \leq G$.

5) The natural map $\phi : G \to G/N$ given by $g \mapsto gN$ is a homomorphism.

6) First Homomorphism Theorem: If $\phi : G \to H$ is a group homomorphism. Then $G/\ker\phi \cong \mathrm{Im}\,\phi$.

PROOF STRATEGY:

The proof is a straightforward application of the First Isomorphism Theorem. Once we get the pieces defined properly, it is immediate. We will show that $\frac{HN}{N}$ is a factor group and invent a homomorphism from H to it that has $H \cap N$ as its kernel.

PROOF:

Draw a baseball diamond. Put $H \cap N$ at home, N at first, HN at second, and H at third. While you're at it, put G in center field. This is the classical "Diamond Isomorphism" proof diagram. We have the following inclusion relationships: $G \geq HN \geq N \geq H \cap N$ and $G \geq HN \geq H \geq H \cap N$. The theorem states that second base mod first base is isomorphic to third base mod home. OK..I'll lose the baseball metaphor now.

First we will attack the issue of legitimizing $\frac{HN}{N}$. We need to show $N \lhd HN$, or equivalently, $(hn)n_1(hn)^{-1} \in N$ for arbitrary $h \in H$ and $n, n_1 \in N$. Now $hn_1h^{-1} \in N$ because $N \lhd G$, hence $gn_1g^{-1} \in N$ for all $g \in G$...and every h is a g. This is a funny situation. Think about it...there is no containment relationship between H and N assumed. It could be that $H \cap N = G$ or $H \cap N = e_G$, or anything in between. We are going to use the normality condition without N being a corresponding normal subgroup of H. OK...back to $(hn)n_1(hn)^{-1} = h[nn_1n^{-1}]h^{-1} = hn_2h^{-1} \in N$. So indeed $N \lhd HN$ and the factor group $\frac{HN}{N}$ makes sense.

Perhaps you've noticed that the homomorphisms that work in the proofs of theorems of this type are pretty simple. Generally speaking, you would expect this, since the assumptions of the theorems are modest and there is no obvious reason to complicate things. But, of course, we speak with the benefit of hindsight. Anyway, here's another. Start with the natural homomorphism $\phi : G \to G/N$ given by $g \mapsto gN$. Now restrict it to H, so we have $\phi|_H : H \to G/N$. We're lazy (but well-meaning), so we try not to do any extra work in our proofs, and by invoking the natural map (known to be a homomorphism), we don't have to show that $\phi|_H$ is a homomorphism, since the restriction inherits that property.

Now what is the kernel of $\phi|_H$? Well, $\ker\phi|_H = \{h \in H : hN = N\}$, since N is the identity for G/N. By the elementary rules for cosets, whenever $hN = N$, we know $h \in N$, hence $\ker\phi|_H = H \cap N$.

OK...what is the image of $\phi|_H$? It is pretty clear that the restriction cuts out all gN where $g \notin H$, and leaves only cosets of the form hN for $h \in H$. But

$\{hN : h \in H\} = \{hnN : h \in H, n \in N\} = \frac{HN}{N}$.

Finally, all of the pieces are available to plug into the *First* Isomorphism Theorem: $H/\ker\phi|_H \cong \operatorname{Im}\phi|_H$ in this context yields $\frac{H}{H \cap N} \cong \frac{HN}{N}$, as we intended.

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