JUCSQ

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①

Th^m's about homomorphisms: Let $\phi : G \to H$, $g \in G$, $h \in H$

1) $\phi(e_G) = e_H$

$\phi(e_G \cdot e_G) = \boxed{\phi(e_G)} \cdot \phi(e_G)$

$\phi(e_G) =$ this acts as e_H

2) $\phi(g^n) = [\phi(g)]^n$

Base case: $\phi(g^2) = \phi(g \cdot g) \overset{\text{def}}{=} \phi(g) \cdot \phi(g)$

Assume true for $n-1$: $\phi(g^{n-1}) = [\phi(g)]^{n-1}$

Want to show $\phi(g^n) = [\phi(g)]^n$

We have $\phi(g^{n-1} \cdot g) = [\phi(g)]^{n-1} \cdot \phi(g)$

$\phi(g^n) = [\phi(g)]^n$ ∎

(2)

aH=H then $a \in H$ — coset result

3) $\ker\phi \leq G$

Let $a, b \in \ker\phi$ ... this means $\phi(a) = \phi(b) = e_H$

$\phi(b^{-1}) = e_H^{-1} = e_H$ so $b^{-1} \in \ker\phi$

So we ask... is $ab^{-1} \in \ker\phi$

$$\phi(ab^{-1}) = \phi(a)\cdot\phi(b^{-1}) = \underset{e_H}{\phi(a)}\cdot\underset{e_H}{[\phi(b)]^{-1}} = e_H$$

This belongs to e_H,

so $ab^{-1} \in \ker\phi \overset{\text{1 step}}{\Rightarrow} \ker\phi \leq G$ ∎

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4) $\phi(a) = \phi(b)$ iff $a\ker\phi = b\ker\phi$

($\Rightarrow$) Assume $\phi(a) = \phi(b)$ Then...

True
$$\begin{cases} \phi(a)\cdot\phi(b^{-1}) = e_H = \phi(ab^{-1}) = e_H \\ ab^{-1} \in \ker\phi \end{cases}$$

$\phi(b^{-1})\phi(a) = e_H \Rightarrow b^{-1}a \in \ker\phi$

$a \in b\ker\phi \Rightarrow a\ker\phi = b\ker\phi$

($\Leftarrow$)

Properties of Subgroups under Homomorphisms

Def'n If ϕ is a homo'm $\phi: G \to G$ it is called an endomorphism

Define $\phi: G \to \bar{G}$

1) If $H \leq G$ then $\phi(H) \leq \bar{G}$

Let $a, b \in \phi(H)$ ... is $b^{-1} \in \phi(H)$?

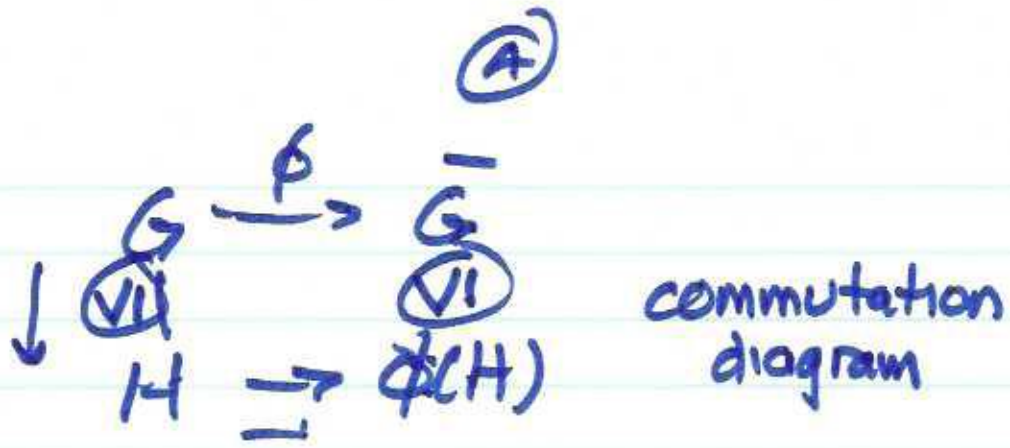
Consider ab^{-1} is it an element $\phi(H)$ of $\bar{G}$

$\phi^{-1}(a)\phi^{-1}(b) \in H$ so

$\phi^{-1}(a)\phi^{-1}(b^{-1}) \in H$ implies

$ab^{-1} = \phi(H)$. So by 1-step test

$\phi(H) \leq \bar{G}$.



2) If H is abelian then $\phi(H)$ is abelian

Let $a, b \in H$ so $ab = ba$ then

by O.P. $\phi(ab) = \phi(a)\phi(b)$ in $\bar{G}$

$\phi(ba) = \phi(b)\phi(a)$

in G

3) If $H \triangleleft G$, then $\phi(H) \triangleleft \bar{G}$ $\phi(G)$

may not be surjective

$G \to \bar{G}$, codomain

$\to \phi(G)$, range

$H \to \phi(H)$ ⇒ normality

$H \triangleleft G$

(5)

Want to show $\phi(H) \triangleleft \phi(G)$ (normal)

Normality test: suppose $N \triangleleft G$, then

Aside $\forall g \in G$, $gNg^{-1} = N$ as sets

$gn_1g^{-1} = n_2$ as elements

Pf:

Take $\phi(h) \in \phi(H)$, pick $\phi(g) \in \phi(G)$

note $\phi(g^{-1}) \in \phi(G)$

Form $\phi(g)\phi(h)\phi(g^{-1}) \in ?$

by O.P. $\phi(ghg^{-1})$ ← same

$\phi(h') \in \phi(H)$

so $\phi(H) \triangleleft \phi(G)$ ■

Given $\phi : G \to \bar{G}$

Claim: $\ker\phi \trianglelefteq G$. (so $G/\ker\phi$ exists)

Let $k \in \ker\phi$ and pick any $g \in G$.

form gkg^{-1} and determine if it is back in $\ker\phi$.

Apply $\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1})$

$\phi(k) \to e_{\bar{G}}$

$= \phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = e_G$

So $gkg^{-1} \in \ker\phi$ i.e. $\ker\phi \trianglelefteq G$ ■

Terminology: "pullback" of an element wrt a map ϕ = ϕ^{-1}(element)

pre-image

Emmy Noether (Nöter)

1st Isom Thm:

Let ϕ be a group homo^{n} from G to $\bar{G}$.

Then the mapping from $G/\ker\phi \to \bar{G}$

Given by $g\ker\phi \to \phi(g)$ is an

Isom^{y}:

$$G/\ker\phi \cong \phi(G)$$

