4T9KS

10/29
①

G/Z problems:

1) Given G non-abelian of order p^3, p prime
If $|Z(G)| > 1$ show $|Z(G)| = p$

Recall order of a factor group is the order of the group divided by the order of the normal subgroup ... so $\left|\frac{G}{N}\right| = \frac{|G|}{|N|}$.

Suppose $|Z(G)| = p^2$ then what is the order of $\frac{G}{Z(G)}$? $= \frac{p^3}{p^2} = p$

So G/Z is cyclic $\xrightarrow{G/Z}$ G abelian $\rightarrow\leftarrow$

Conclude $|Z(G)| = p$

(2)

2) Given $|G| = pq$ where p, q are primes

Show that $Z(G)$ = either $\{e\}$ or $Z(G) = G$

Suppose G is <u>not</u> centerless i.e. $|Z(G)| > 1$

Assume $|Z(G)| = p$ (mutatis mutandis for q). What is order of $\frac{G}{Z(G)}$ in this case? Order is q. By G/Z theorem

G is abelian (i.e. has order pq)

Group Homomorphisms

Def^{n} A mapping $\phi : G \to H$ is a homo^{m} if ϕ is O.P. (i.e. for $a, b \in G$ we have $\phi(ab) = \phi(a)\phi(b)$

Def^{n} The kernel of a homomorphism is the set $\{x \in G : \phi(x) = e_H\}$

Usually denoted by $\ker \phi$

Ex: 1) Every iso^{m} is a homo^{m}

2) A homo^{m} that is injective is called a monomorphism

3) A homo^{m} that is surjective is called an epimorphism

4) Consider $\phi : GL(2, \mathbb{R}) \to \mathbb{R} - \{0\}$

Is the det map O.P. ?

$\det(AB) \overset{?}{=} \det A \cdot \det B$ ✓

So... $\phi = \det(\ )$ is an epimorphism

5) What is kernel of this map

(all 2×2 matrices which have det 1)

$SL(2,\mathbb{R})$

We would write $\ker\phi = SL(2,\mathbb{R})$

5) Characterize map $abs(x) = |x|$, $x \in \mathbb{R}$

$abs(xy) = abs(x) \cdot abs(y)$

so $abs(\cdot)$ is a homo^{my}

6) Let $\mathbb{R}[x]$ What is the classification of the derivative map.

Given $P(x) \in \mathbb{R}[x]$, $\phi(P(x)) = P'(x)$ ← operation = addition

This is sum/diff rule in calc.

What is $\ker\phi$? $\ker\phi = \{\text{constant poly's}\}$

+) Is $\phi(x) = x^2$ for $x \in \mathbb{R}_{0+}$

$\phi(x+y) = (x+y)^2 = x^2+y^2+2xy$

8) In $\mathbb{Z}_2$, this would be OK

Likewise if we defined $\phi(x) = x^p$

and $\phi: \mathbb{Z}_p \to \mathbb{Z}_p$

$$(x+y)^p = x^p + y^p \pmod p$$

$$(x+y)^p = x^p + px^{p-1}y + \binom{p}{2}x^{p-2}y^2 + \cdots + y^p$$

Check: does $\binom{p}{k} = 0 \bmod p$ for $k = \phi$ to $p-1$

$$\frac{p!}{(p-k)!\,k!}$$

$\phi : G \to H$

Properties of Homomorphisms (of groups)

1) Identities are preserved

$\phi(e_G) = e_H$

Pf: $\phi(e_G \cdot g) = \phi(e_G)\phi(g)$

$\downarrow$

$\phi(g) = \phi(e_G)\phi(g)$

↑ must be identity in H

2) $\phi(g^n) = [\phi(g)]^n$

$\phi(g^2) = \phi(g)\phi(g) = [\phi(g)]^2$ base case

$\phi(g^n) = \phi(g^{n-1} \cdot g) = \phi(g)^{n-1} \cdot \phi(g) =$

$[\phi(g)]^n$ induction step

hence true for all $n \in \mathbb{N}$

3) $\ker \phi \leq G$

Take $x, y \in \ker \phi$

Then $\phi(x) = \{e_H\}$, $\phi(y) = \{e_H\}$

$\phi(y^{-1}) = [\phi(y)]^{-1}$

Is $xy^{-1} \in \ker \phi$? In order for this to be true, $\phi(xy^{-1}) = \{e_H\}$

$\phi(xy^{-1}) = \phi(x)\phi(y^{-1}) = \phi(x)[\phi(y)]^{-1} =$

$e_H \cdot [e_H]^{-1} = e_H$ ... so apparently

$\phi(xy^{-1}) \in \ker \phi$ hence $\ker \phi \leq G$. ∎

