# SN8EM

10/22



$G/Z$ Th$^{m}$:

If for any group $G$, $G/Z$ is cyclic, then $G$ is abelian. ($Z$ is the center of $G$) i.e. $G = Z(G)$

Pf: Since $G/Z$ is cyclic, there is some generator, say $G/Z = \langle gZ(G) \rangle$. So given an arbitrary $a \in G$, there exists some $aZ(G) = [gZ(G)]^i = g^i Z(G)$

$a = g^i z_1$, where $z \in Z(G)$. Now take $bG(Z)$ & do the same. So $b = g^j z_2$

Now consider $ab = (g^i z_1)(g^j z_2) =$

$(g^i z_1)(z_2 g^j) = g^i (z_1 z_2) g^j =$

$g^i g^j (z_1 z_2) = g^j g^i (z_2 z_1) =$

$(g^j z_2)(g^i z_1) = ba$ so $G = Z(G)$

[If $G/Z$ is cyclic, $G$ is abelian]

contra If $G$ is non-abelian, then $G/Z$ not cyclic.

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(67) If $G$ is non-abelian, show that $Aut(G)$ is not cyclic / neither is $G$

Suppose $\phi : G \to G \in Aut(G)$

$\phi(G)$ is not abelian

$\phi_x := xGx^{-1}$ $x \in G$

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Cauchy's Weak Converse of Lagrange's Th$^{m}$

Th$^{m}$ Let $G$ be finite group. Let $p$ be a prime such that $p \mid |G|$, then $G$ has an element of order $p$. (hence a subgroup of order $p$).

Proof (by induction)

Clearly true for $|G| = 2$.

Now assume that statement is true for all abelian groups of order $< |G|$. Note: $G$ has elements of prime order. Why?

Say $x \in G$ has order $m$. i.e. $x^m = e_G$.

then either $m$ is prime or $m = nq$ where $q$ is a prime factor $m$. Then $x^{nq} = (x^n)^q = e_G$ but this means $x^n$ has prime order $q$. So pick element $x$ with prime power order. $p$ Construct factor group

$\bar{G} = \frac{G}{\langle x \rangle}$ note $\bar{G}$ is abelian and

$p \mid |G|$ so $|\bar{G}| = \frac{|G|}{p} < |G|$ so $G$ does as well ∎

$Z_n$ is finite and abelian, so Cauchy's theorem applies. Then if $p$ is a prime and $p \mid n$, $Z_n$ has an element of order $p$.

$Z_{105}$ – does it have elements of order 3, 5, 7? $3 \cdot 5 \cdot 7 = 105$

How many subgroups are there in $Z_{105}$? order 3, 5, 7, 35, 21, 15

$\{0, 1, 2, 3, 4\}$

External Direct Product of $G_1$ and $G_2$

is set $\{(g_1, g_2) : g_1 \in G_1 \text{ & } g_2 \in G_2\}$

group operation is component wise

there does not need to be any relationship between $G_1$ & $G_2$ $\quad G_1 \oplus G_2$

Can generalize to $G_1 \times G_2 \times G_3 \times \cdots \times G_n$

Internal Direct Product (internal to a single group).

G is the internal direct product of $H \leq G$ and $K \leq G$ written $G = H \times K$, if $H \trianglelefteq G$, $K \trianglelefteq G$, and $H \cap K = \{e_G\}$.

$D_n$ - n-gon

Ex: $D_6$ is the symmetry group of 6-gon

Let F be any reflection and $R_k$ be a rotation thru k degrees

$D_6 = \{R_0, R_{180}\} \times \{R_0, R_{120}, R_{240}, F, R_{120}F, R_{240}F\}$

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What can go wrong.

Let $G = S_3$ $|S_3| = 3! = 6$

$H = \langle (123) \rangle$ and $K = \langle (12) \rangle$

(231) (21)

(312)

↑ not normal subgroup

$H \cap K = \{1\}$

Internal Direct Product of $H_1, H_2, H_3, \dots H_n$ :

finite collection of normal subgroups of $G$.

Then $G$ is the internal direct product if

(i) $G = \{h_1 h_2 \cdots h_n : h_i \in H_i\}$

(ii) $(H_1 H_2 \cdots H_i) \cap H_{i+1} = \{e_G\}$ $\quad i = 2 \dots n$