R42JJ

10/20

Normal Subgroups

- identified by E. Galois... led to solution of quintic unsolvability problem

Def^{n} Suppose $H \leq G$. If $gHg^{-1} = H$, then we say "H is normal in G." Usually, if we know H is normal, we denote it by N, to make it crystal clear.

Not: N is normal in G "$N \trianglelefteq G$"

$gN = Ng$ or $gNg^{-1} \in N$

~~$gn = ng$~~ $gn_1 = n_2 g$

Th^{m} any subgroup $H < G$ with index 2 in G is automatically normal

2 cosets

H, gH $g \notin H$ Ask where would Hg be?

Conclude $gH = Hg$ i.e. $H \trianglelefteq G$.

Show $A_n \trianglelefteq S_n$

(*) If $[G:H] = 2$, then $H \trianglelefteq G$

Consider a unique subgroup H of G.

An auto^m $\phi : G \to G$ of form $gHg^{-1} < G$

(*) Forces $H = gHg^{-1}$ normality

Given $Z(G)$ is $Z(G) \trianglelefteq G$

$gZ(G) \overset{?}{=} Z(G)g$

same element

$gZ_1 = Z_1 g$

commutes w/ all elements

★ Thm Let $N \trianglelefteq G$ and $H \leq G$, then

$HN \leq G$.

Recall $HN = \{hn : h \in H, n \in N\}$

Show $HN \leq G$:

i) $ee = e \in HN$ identity is present

Now take $a \in HN$, $a = h_1k_1$

$b = h_2k_2$, claim $\exists h' \in H$ such that

$ab^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1(k_1k_2^{-1})h_2^{-1} =$

$(h_1h')(k_1k_2^{-1}) \in HN$ so $ab^{-1} \in HN$ ∎

★ If G is abelian and $H \leq G$, then

$H \trianglelefteq G$. $\quad gHg^{-1} = H$

↑ any subgp

④

Find a non-abelian subgroup of D_8

Define $H = \{R_0, R_{90}, R_{180}, R_{270}\}$

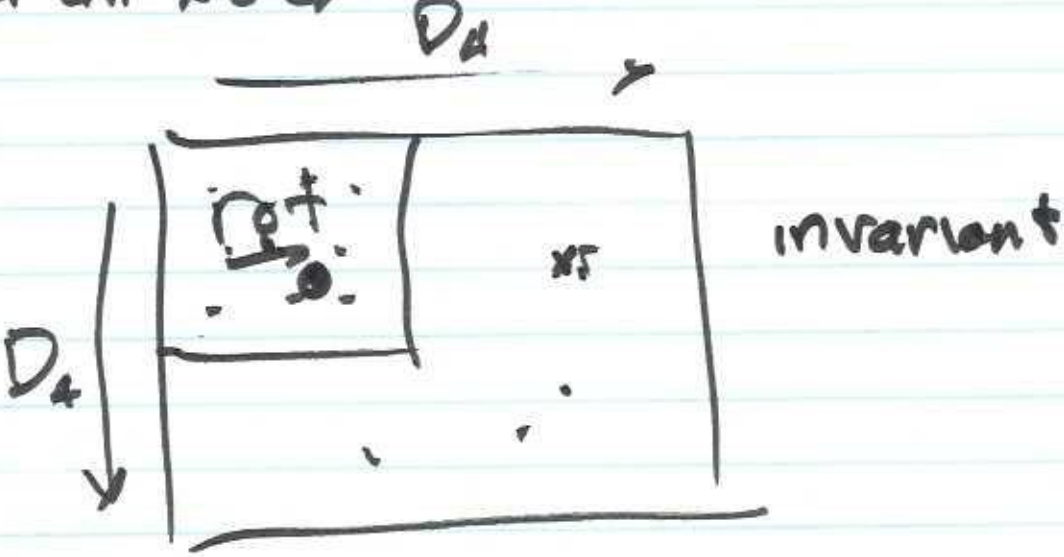
$K = \{R_0, F\}$ F any flip

Then $HK = \{R_0, R_{90}, R_{180}, R_{270}, \ldots$

$\ldots R_0F, R_{90}F, R_{180}F, R_{270}F\}$

Normality Test:

A subgp $H \leq G$ is normal iff $xHx^{-1} \subseteq H$ for all $x \in G$



($\Rightarrow$) Assume $H \trianglelefteq G$. Then by def'n $x \in G, h \in H, \exists h' \in H \ni xh = h'x$

Then $xhx^{-1} = h'$ so since $h' \in H$, $xHx \subseteq Hx$ so $xhx^{-1} \in H$ or $xHx^{-1} \in H$

($\Leftarrow$) Suppose $xHx^{-1} \subseteq H$, $\forall x \in G$. let $x = a$ then $aHa^{-1} \subseteq H$ which is equivalent to $aH \subseteq Ha$. Now let $x = a^{-1}$, then $a^{-1}Ha \subseteq H$ which is the same as $a^{-1}H \subseteq Ha^{-1} \Rightarrow Ha \subseteq aH$ So $aH = Ha$, i.e. H is normal in G. $H \trianglelefteq G$. ∎

Factor Groups

Let G be a group and $H \trianglelefteq G$. The set

$\{ G/H : aH \text{ where } a \in G \}$ subject to

operation $(aH)(bH) = abH$

this operation requires it to be normal to be well-defined

$\phi : G/H \times G/H \to G/H$

Must show if $a'H = aH$ and $b'H = bH$

then need $aHbH = a'Hb'H$ or

$abH = a'b'H$

(7)

$4\mathbb{Z} = \{0, \pm 4, \pm 8, \cdots\}$

Construct a finite group from $\mathbb{Z}$

by noting $\left\{\frac{\mathbb{Z}}{4\mathbb{Z}}\right\} = \{0+4\mathbb{Z}, 1+4\mathbb{Z},$

$2+4\mathbb{Z}, 3+4\mathbb{Z}\}$

$\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\{0+4\mathbb{Z}\} = \{0, 4, 8, 12, 16, \cdots\}$

$\{1+4\mathbb{Z}\} = \{1, 5, 9, 13, 17, \cdots\}$

$\{2+4\mathbb{Z}\} = \{2, 6, 10, 14, 18, \cdots\}$

$\{3+4\mathbb{Z}\} = \{3, 7, 11, 15, 19, \cdots\}$

	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$			
$1+4\mathbb{Z}$	$1+4\mathbb{Z}$			
$2+4\mathbb{Z}$	$2+4\mathbb{Z}$			$1+4\mathbb{Z}$
$3+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$		

We have noted that $\frac{\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}_4$

Is it true that $\frac{\mathbb{Z}}{n\mathbb{Z}} \cong \mathbb{Z}_n$

Notation: what does $|aH|$ mean

$ah \in aH$

(*) The G/Z Theorem