

①

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Given groups $G_1, G_2, \dots, G_n, \dots$ why is

$\bigoplus_{i=1}^{\infty} G_i$ a group? Typical element:
 $(g_1, g_2, \dots, g_n, \dots)$

Proof: Closure - each component is closed
 so group operation must be closed

Associativity - each component is associative

Identity - $(e_{G_1}, e_{G_2}, \dots, e_{G_n}, \dots)$

Inverse - $(g_1^{-1}, g_2^{-1}, \dots, g_n^{-1}, \dots)$

$\therefore \bigoplus_{i=1}^{\infty} G_i$ is a group. ■

We showed $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ had following Cayley T.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	this is an abelian group
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$	
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$	
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$	

②

V_4 is the "Klein group"

$$\text{id: } e, a, b, c \quad a^2 = b^2 = c^2 = e$$

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	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$\phi \left\{ \begin{array}{l} e \mapsto (0, 0) \\ a \mapsto (1, 0) \\ b \mapsto (0, 1) \\ c \mapsto (1, 1) \end{array} \right. \left| \begin{array}{l} \text{bijectivity - established by} \\ \text{def'n of } \phi \end{array} \right.$$

$$\phi(x+y) = \phi(x) + \phi(y)$$

$$x, y \in V$$

$$\phi(b+c) \stackrel{?}{=} \phi(b) + \phi(c)$$

a
in V_4

$(0, 1) + (1, 1) = (1, 0)$
in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

a

③

Criterion for Cyclicity of Direct Sums

So... given $G \oplus H$ then G, H cyclic

$G \oplus H$ is cyclic iff $|G|$ & $|H|$ are relatively prime

Aside: $\text{lcm}(a, b) := [a, b]$

$\text{gcd}(a, b) := (a, b)$

$\text{gcd}(a, b) = 1$ means relatively prime

Proof: How many elements are in $G \oplus H$?

$$|G| = m \quad |H| = n \Rightarrow |G \oplus H| = mn$$

Suppose $G = \langle g \rangle$ and $H = \langle h \rangle$

$$\text{so... } g^m = e_G \text{ and } h^n = e_H$$

Also suppose $(m, n) = 1$ (rel. prime)

So... what is $\text{lcm}(m, n) = [m, n] = ?$

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$$\rightarrow m \cdot n = \underbrace{[m, n]}_{\text{lcm}} \cdot \underbrace{(m, n)}_{\text{gcd}}$$

$$\quad \quad \quad \textcircled{mn} \quad \quad \textcircled{1}$$

Then we have $|(g, h)| = mn$ which means $G \oplus H$ is cyclic (w/ generator (g, h))

Corollary: Generalize to $G_1 \oplus G_2 \oplus G_3 \cdots \oplus G_n$

$$G = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$$

$$\text{Corollary: } \mathbb{Z}_{n_1 n_2 \cdots n_k} \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$$

whenever $(n_i, n_j) = 1$ if $i \neq j$

~~IF~~ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_4$

T $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12}$ because

$$\boxed{(3, 4) = 1}$$

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$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{30}$$

Prob's: ① Is $\mathbb{Z} \oplus \mathbb{Z}$ cyclic?

Suppose $\langle (a, b) \rangle = \mathbb{Z} \oplus \mathbb{Z}$

③ Let G be group w/ id e_G and

H be a group w/ id e_H . Show that

$$\underline{G \cong G \oplus \{e_H\}} \text{ and } H \cong \{e_G\} \oplus H$$

$$\underline{(g, e_H) \mapsto g}$$

④ Show $G \oplus H$ is abelian iff G, H abelian

Suppose $\overset{\text{FSOC}}{(g_1, h_1) \cdot (g_2, h_2) \neq (g_2, h_2) \cdot (g_1, h_1)}$

either $g_1, g_2 \neq g_2, g_1$ or $h_1, h_2 \neq h_2, h_1$ } ~~✗~~

⑥

⑧ Is $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{27}$ NO

⑨ Give an example of an abelian group of order 12 that has two subgroups of order 6

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong (a, b, c)$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_6 \quad \underline{(0, a)} \quad a \in \mathbb{Z}_6$$

$$\underline{\mathbb{Z}_4 \oplus \mathbb{Z}_3} \cong \mathbb{Z}_{12} \quad (a, b) \quad a \in \{0, 1, 2, 3\}$$

$$b \in \{0, 1, 2\}$$