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Algebraic vs. Transcendental Extensions:

An element α is transcendental over field F if it is not a root of any polynomial w/ coefficients in F. If we form $E = F(\alpha)$, E is a transcendental extension (of F).

An element α is algebraic over field F if it is a root of some polynomial (arbitrary degree) w/ coefficients in F. Then $E = F(\alpha)$ is an algebraic extensions.

Gelfond Th^{m}

If $\alpha \neq 0, 1$ and β is irrational then α^{β} is transcendental (over $\mathbb{Q}$)

$$\left(2^{\sqrt{2}}\right)^{\sqrt{2}} = 2^{(\sqrt{2})^2} = 2^2 = 4 \, !$$

If α is algebraic over $\mathbb{Q}$ must $\sqrt[n]{\alpha}$ be algebraic? Consider square roots.

If α is a root of $a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$ " $p(x)$ "

Look @ $a_n x^{2n} + a_{n-1} x^{2(n-1)} + \cdots a_1 x^2 + a_0$.

So $\sqrt{\alpha}$ is root of $\hat{p}(x)$ So yes for integer roots.

How about α^2

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$ α is zero

Scheme:

all even

$$\tilde{p}(x) = \left[a_n x^n + a_{n-2} x^{n-2} + \cdots + a_0\right] - x\left[a_{n-1} x^{n-1} + \cdots a_1 x\right]$$

$$\tilde{p}'(x) = \left[a_n x^n + \cdots + [illegible] + a_0\right] + x\left[a_{n-1} x^{n-1} + \cdots a_1 x\right]$$

$$f(x) = \tilde{p}(x) \cdot \tilde{\tilde{p}}(x) \Rightarrow f(\alpha) = 0$$

$$\left[a_n x^n + a_{n-2} x^{n-2} + \cdots a_0\right]^2 - x^2 \left[a_{n-1} x^{n-1} + \cdots a_1 x\right]^2$$

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1 Double Cube

Can assume cube has volume 1, so side is 1

To double cube, need side $= \sqrt[3]{2}$

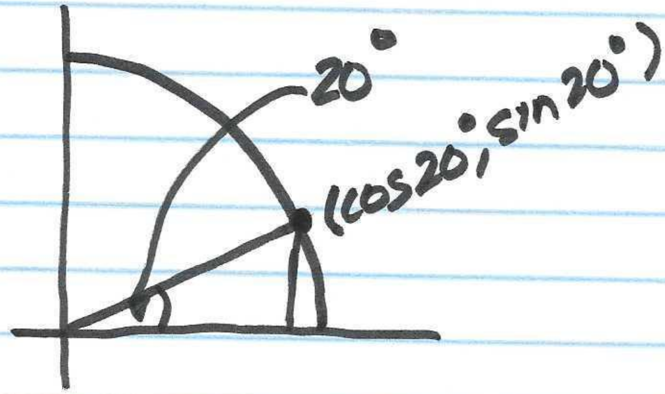
Field of constructible numbers. $x^3 - 2 = 0$

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{\alpha}) \subset \mathbb{Q}(\sqrt{\alpha}, \sqrt[4]{\alpha}) \subset \cdots$$

$$[\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}] = 2$$

$$[\tilde{\mathbb{Q}} : \mathbb{Q}] = 2^k$$

④



Consider poly which has cos 20° as root.

$$\cos 60^\circ = \frac{1}{2} = \cos(3 \cdot 20^\circ)$$

$$\cos 3\theta = \cos(\theta + 2\theta)$$

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$= \cos^2\theta - \sin^2\theta = \underbrace{\cos^2\theta - (1-\cos^2\theta)}_{2\cos^2\theta - 1}$$

$$\cos(2\theta + \theta) =$$

$$\rightarrow (2\cos^2\theta - 1)\cos\theta - \sin^2 2\theta \cdot \sin\theta$$

$$2\cos^3\theta - \cos\theta - (1 - \cos^2 2\theta)(\sin\theta)$$

$$2\cos^3\theta - \cos\theta - (1 - \cos^2 2\theta \cdot)\sin\theta$$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\theta = 20^\circ$

$$\frac{1}{2} = 4\cos^3 20^\circ - 3\cos 20^\circ$$

$x = \cos 20^\circ$

$$\frac{1}{2} = 4x^3 - 3x \Rightarrow$$

$$\boxed{8x^3 - 6x - 1 = 0} \leftarrow \text{irreducable?}$$

let $y = x+1 \quad x = y-1$

$$8(y^3 - 3y^2 + 3y - 1) - 6(y-1) - 1 = 0$$

$$8y^3 - 24y^2 + 24y - 8 - 6y + 6 - 1 = 0$$

$$8y^3 - 24y^2 + 18y - 3 = 0$$

$p = 3$ in Eisenstein $\Rightarrow$ poly is irreducible