4-17

Given an extension, $F(\alpha)$, there is a unique monic polynomial of minimal degree for which the evaluation @ α is zero.

$f(x)$ over $F(\alpha)$ - this is called the "minimal polynomial."

① $\mathbb{Q}(\sqrt{2}) \rightarrow x^2-2=0$

② $\mathbb{Q}(\sqrt{2},\sqrt{3})$ then what is the minimal polynomial for $\sqrt{2}+\sqrt{3}$?

$x-(\sqrt{2}+\sqrt{3})=0$

$x=\sqrt{2}+\sqrt{3}$

$x^2= 2+2\sqrt{2}\sqrt{3}+3 = 5+2\sqrt{6}$

$x^2-5 = 2\sqrt{6}$

$x^4-10x^2+25=24$

$\boxed{x^4-10x^2+1=0}$ ← irr.

②

Find minimal poly for $x = \sqrt{2} + \sqrt[3]{2} =$

$(x - \sqrt{2}) = \sqrt[3]{2}$

$$x^3 - 3\sqrt{2}\,x^2 + 3\underbrace{(\sqrt{2})^2}_{2}x - \underbrace{(\sqrt{2})^3}_{2\sqrt{2}} = 2$$

$$x^3 - 3\sqrt{2}\,x^2 + 6x - 2\sqrt{2} = 2$$

$$x^3 + \sqrt{2}(-3x^2 - 2) + 6x - 2 = 0$$

$$x^3 + 6x - 2 = (3x^2+2)\sqrt{2}$$

$$(x^3+6x)^2 - 2(x^3+6x) + 4 = (3x^2+2)^2 \cdot 2$$

$$x^6 + 12x^4 + 36x^2 - 2x^3 - 12x + 4 =$$

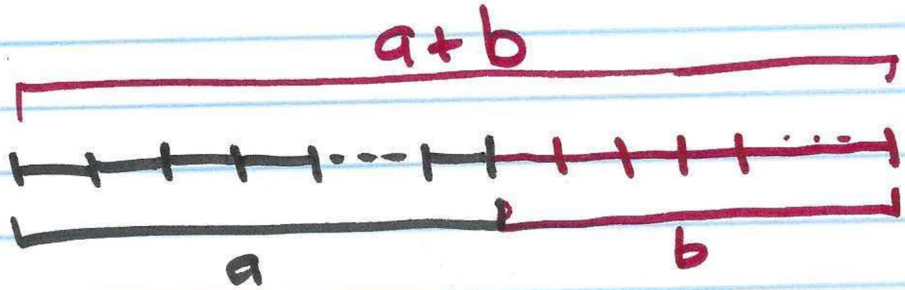
$$(9x^4 + 12x^2 + 4)2$$

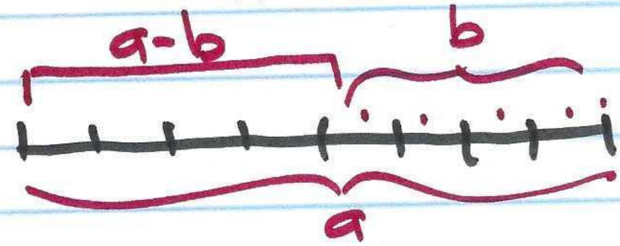
$$\cancel{x^6} + 12\cancel{x^4} + 3\cancel{6x^2} - \cancel{2x^3} - 12x + 4 - 1\cancel{8x^4} - \cancel{24}x^2 - 8 = 0$$

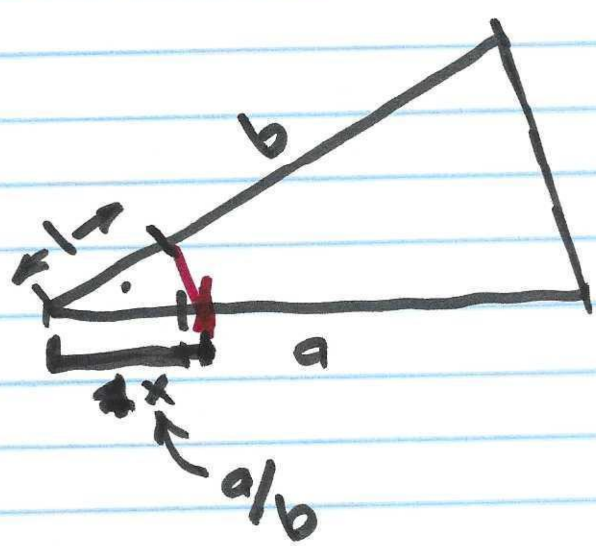
$$\star\ x^6 - 6x^4 - 2x^3 + 12x^2 - 12x - 4 = 0$$

Field of Constructible Numbers:

How do you construct $a \pm b$



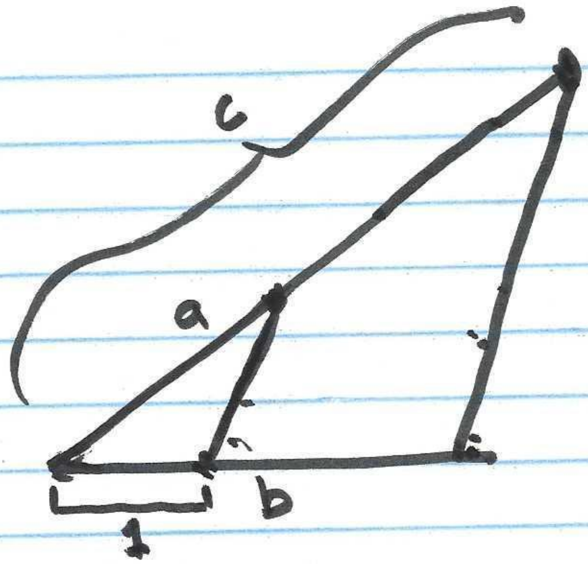




$$\frac{x}{1} = \frac{a}{b}$$

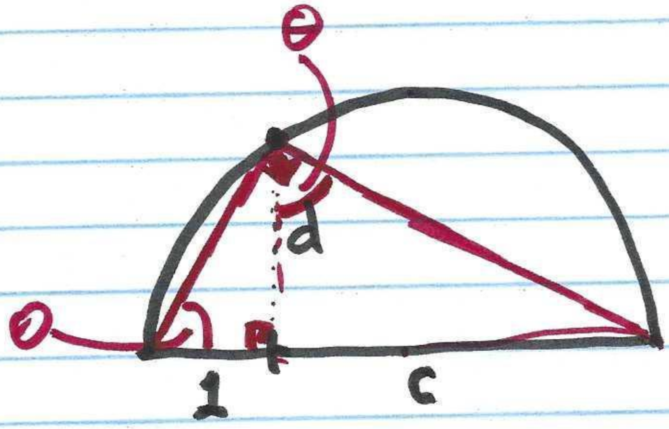
so $x = \frac{a}{b}$

4



$$\frac{a}{1} = \frac{c}{b}$$

$$ab = c$$



$$\frac{c}{d} = \tan\theta$$

$$\frac{d}{1} = \tan\theta$$

$$\frac{c}{d} = \frac{d}{1} \Rightarrow$$

$$c = d^2$$

$$\sqrt{c} = d$$

$x, y \in F$

Let $\{(x,y) \in \mathbb{R}^2\}$ be the <u>plane of F</u>

$\mathbb{F} \subseteq \mathbb{R}$

(i) $\mathbb{F} \neq \emptyset$

(ii) $\forall x, y \in F \Rightarrow x - y \in F$

(iii) $\forall x, y \in F \Rightarrow \frac{x}{y} \in F$

(i) A line in F connects two points in plane of F

(ii) A circle in F has a center & radius in F

Line has equation $ax + by + c = 0$

Circle has equation $x^2 + y^2 + ax + by + c = 0$

What points in $\mathbb{R}$ can be gotten from points in F by using ruler/compass.

i) 2 lines intersect

ii) 2 circles intersect

iii) line & circle intersect

Case(i) $\ell_1 : ax + by + c = 0$

$\ell_2 : dx + ey + f = 0$

$$\begin{cases} dax + dby = -cd \\ dax + eay = -af \end{cases}$$

$y = \dfrac{af - cd}{db - ea}$ & similarly for x

$\ell_1 \cap \ell_2$ is point in $\mathbb{F}$

Case(ii) $C_1 \cap C_1$ is point in $\mathbb{F}$

Case(iii) $C_1 \cap \ell_1$ is point in $\mathbb{F}$

So constructible points in $\mathbb{R}$ are those in $\mathbb{F}$ plus square roots of elements of $\mathbb{F}$.

$\sqrt{2}\ \ \sqrt[4]{2}\ \ \sqrt[8]{2} \cdots \in \mathbb{F}$

(*) $[\mathbb{F}(\alpha) : \mathbb{F}] = 2^n$ some n.