Some problems from Ch 20.

① Show $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$

$\mathbb{Q}(\sqrt{2},\sqrt{3}) \quad \sqrt{2}+\sqrt{3} \Rightarrow \mathbb{Q}(\sqrt{2}+\sqrt{3})$

$3-2 = (\sqrt{3})^2-(\sqrt{2})^2$

$\frac{3-2}{\sqrt{3}+\sqrt{2}} = \sqrt{3}-\sqrt{2} \in$

$(\sqrt{3}+\sqrt{2})+(\sqrt{3}-\sqrt{2}) = 2\sqrt{3} \cdot \frac{1}{2} = \sqrt{3}$

$\sqrt{2}+\sqrt{3}-\sqrt{3} = \sqrt{2}$

$\mathbb{Q}(\sqrt[3]{2},\sqrt[3]{4}) = \mathbb{Q}[\sqrt[3]{2},\sqrt[3]{4}]$

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② Find splitting field of x^3-1 over $\mathbb{Q}$

Express your answer as $\mathbb{Q}(\alpha)$

$x^3+px+q=0 \Rightarrow u,v \ni x=u+v$

$\omega u+\omega^2 v,\ \omega^2 u+\omega v \qquad \omega = \frac{-1+i\sqrt{3}}{2}$

3 complex roots of 1:

$1, \omega, \omega^2$

$\mathbb{Q}(\omega)$.

$$\omega = \frac{-1+i\sqrt{3}}{2}, \quad \omega^2 = \frac{-1-i\sqrt{3}}{2}$$

$$\omega^2 = \frac{1}{4}\left(-1+i\sqrt{3}\right)^2 = \frac{1}{4}\left(1-2i\sqrt{3}-3\right)$$

$$\frac{1}{4}\left(-2-2i\sqrt{3}\right) = \frac{1}{2}\left(1-i\sqrt{3}\right)$$

$$\overset{\omega}{\left(\frac{-1+i\sqrt{3}}{2}\right)} \cdot \overset{\omega^2}{\left(\frac{-1-i\sqrt{3}}{2}\right)} =$$

$$\frac{1}{4}\left(1+3\right) = 1$$

$x^2-2=0$ $\quad x=\sqrt{2} \leftarrow$ "algebraic"

$a_n x^n + \cdots + a_1 x + a_0 = 0$ $\quad$ 1871 $\quad$ 1884

$x = e$ or π

$\lambda = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ $\quad$ Liouville's Number

$.110001$ —— $\overset{24}{\rightarrow} 1 \overset{120}{\longrightarrow} \cdots \rightarrow 1$

$a_n x^n = -a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$

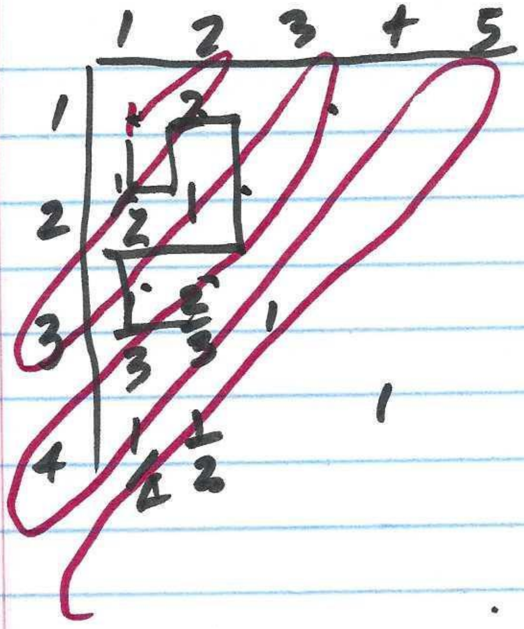
"Ocean of Zeroes"

$\mathbb{Z}_+$

n, not a perfect square

$x^2 - n \quad (x-\sqrt{n})(x+\sqrt{n})$

DONE



$$x - \alpha = 0$$

Given $\frac{m}{n}$, consider

$\alpha = 2^m 3^n$ (5)

$$\{2^m 3^n : m, n \in \mathbb{Z}\} \subset \mathbb{Z}_{\geq 0}$$

α is algebraic, it satisfies an equation

of form $a_n x^n + \cdots a_1 x + a_0 = 0$

$\{P(x) : \text{poly's over } \mathbb{Q}\}$

$$h(p(x)) = \partial p + \sum_{n=0}^{\partial p} |a_n|$$

h is "height"

n	
0	0
1	1
	-1
2	2
	-2
	x
	$-x$
3	3
	-3
	$x+1$
	$x-1$
	$-x+1$
	$-x-1$
	x^2
	$-x^2$
	$2x$
	$-2x$
	$\downarrow$

← each finite

$$e = (1+h)^{\frac{1}{h}} \to \text{take limit as } h \to 0 \qquad \cancel{1^{\infty}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$= \cancel{1 +} 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \to$$

$$\frac{1}{2^n} \qquad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$0 \le a_n \le b_n$$

factorial

powers of two

Then if $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

$$\frac{1}{2^n} > \frac{1}{n!}$$

$$\frac{1}{\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}}} \qquad \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n} \to$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\cancel{s_n = \sum_{n=0}^{k}}$$

$$s_k = \sum_{n=0}^{k} \frac{1}{n!}$$

$$\underbrace{1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}}_{k \text{ terms}} + \cdots \frac{1}{9!}$$

$$0 < e - s_k = \left(\frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \frac{1}{(k+3)!} + \cdots \right.$$

$$= \frac{1}{(k+1)!} \left[1 + \frac{1}{k+2} + \frac{1}{(k+2)(k+3)} + \cdots \right.$$

$k+1$ $k+1$

$$0 < e - s_k <= \frac{1}{(k+1)!} \left[1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{1}{(k+1)^3} + \cdots \right.$$

$$\frac{1}{1 - \frac{1}{k+1}} = \frac{1}{\frac{k+1-1}{k+1}} = \frac{k+1}{k}$$

$$0 < e - s_k < \frac{1}{(k+1)!} \cdot \frac{k+1}{k} = \frac{1}{k \cdot k!}$$

FSOC suppose $e = p/q$

set $k = q$

$$0 < e - s_q < \frac{1}{q \cdot q!}$$

$$0 < \underbrace{\frac{p}{q} \cdot q!}_{\substack{p(q-1)! \\ \in \mathbb{Z}}} - \underbrace{s_q \cdot q!}_{\in \mathbb{Z}} < \frac{1}{q} < 1$$

Find "minimal" polynomial

$\sqrt{2}$ $\quad x^2 - 2 = 0$

$(x - \sqrt{2}) + \sqrt{7} = 0$