Splitting fields exist:

Let F be a field $f(x) \in F[x]$, $f(x) \neq \text{const}$

Then there exists a splitting field E for

$f(x)$.

Pf: By induction on ∂f.

Base Case: Let $f(x)$ be linear. Already

splits over F. - Done

Ind. Hyp. Suppose claim is true for

all poly's of degree $< \partial f$ and all extensions

of F. By Kronecker's Th^{m}, $\exists$ an extension

$E \supset F$ in which f has a root, say r_1.

So in E, $f(x) = (x-r_1)\, g(x)$. $\in E[x]$

Now $\partial g < \partial f$, so by induction, $g(x)$

splits in $K \supset E \supset F$.

So we have shown existence of field $K \supset F$ in which $f(x)$ factors into linear factors i.e. splits.

In K, $f(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$

where $r_i \in K$.

$K = F(r_1, r_2, r_3 \cdots r_n)$

Characterize extension field 2 ways.

$f(x) = x^2 - 2 \in \mathbb{Q}[x]$

BTW $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}]$

$(x-\sqrt{2})(x+\sqrt{2})$

$K = \mathbb{Q}(\sqrt{2})$

$K = \dfrac{\mathbb{Q}[x]}{\langle x^2 - 2 \rangle}$ $\cong$

Consider $\underline{x^6 - 2}$ - does not split in $\mathbb{Q}(2^{1/6})$

basis - $\{1, 2^{1/6}, 2^{1/3}, 2^{1/2}, 2^{2/3}, 2^{5/6}\}$

dim K over $\mathbb{Q}$ is $\underline{\underline{\text{six}}}$ $\quad [K : \mathbb{Q}] = 6$

General Rule for splitting field of $x^n - a$ over $\mathbb{Q}$

$\{a^{1/n}, \omega a^{1/n}, \omega^2 a^{1/n}, \ldots \omega^{n-1} a^{1/n}\}$

Given $ax^2 + bx + c \in \mathbb{Q}[x]$ what is splitting field?

$\underline{\Delta = b^2 - 4ac}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$\Delta < 0 \quad \mathbb{Q}(i, \sqrt{r})$

$$a x^3 + \underline{b x^2} + c x + d = 0$$

$$y = x - \frac{b}{3a}$$ von Tschirnhaus

$$x^{n-1}, x^{n-2}$$

Jerrald $x^{n-1}, x^{n-2}, x^{n-3}$

$\underline{x^5 + a x + b = 0}$ <u>Bring-Jerrald</u>

$$\to x^3 + p x + q = 0$$ (depressed)

Let $x = \underline{u + v}$

$$u^3 + 3u^2 v + 3 u v^2 + v^3 + p(u+v) + q = 0$$

$$u^3 + 3uv(\underline{u+v}) + v^3 + p(\underline{u+v}) + q = 0$$

$$u^3 + \underbrace{(3uv + p)}_{=0}(u+v) + v^3 + q = 0$$

$$u^3 + v^3 + q = 0$$

$$uv = -\frac{p}{3} \Rightarrow u = -\frac{p}{3v}$$

$$-\frac{p^3}{3v^3} + v^3 + q = 0$$

$$-p^3 + 3(v^3)^2 + q(3v^3) = 0$$

resolvent $$(v^3)^2 + qv^3 - \frac{p^3}{27} = 0$$

$$1v^3 = \frac{-q \pm \sqrt{\frac{4q^2}{4} + \frac{4p^3}{27}}}{2}$$

$$1v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}}$$

$$v = \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}$$

$p = 6$ $\quad q = -20$ $\quad 6$

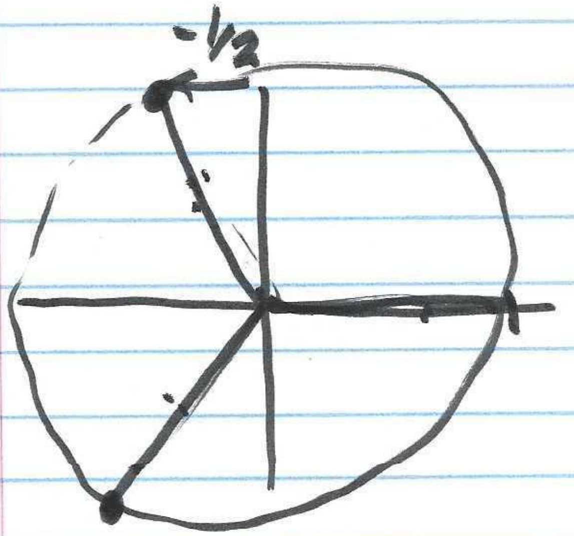
$$x^3 + 6x - 20 = 0 \qquad x = 2$$

$$\omega^3 u = \sqrt[3]{10 + \sqrt{100 + 8}} = 2.732$$

$$\omega^3 v = \sqrt[3]{10 - \sqrt{108}} = -0.732$$

$$2.0$$

$$\sqrt[3]{1} = \underset{\omega^3}{1}, \ \underset{\omega}{\frac{-1 + i\sqrt{3}}{2}}, \ \underset{\omega^2}{\frac{-1 - i\sqrt{3}}{2}}$$



$$\left.\begin{array}{l} u + v \\ \omega u + \omega v \\ \omega^2 u + \omega^2 v \\ \omega u + \omega^2 v \\ \omega^2 u + \omega v \end{array}\right\} 5$$

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

$$(x^2 + cx + 1)(x^2 + dx + 1) = 0$$

Zeroes of Irreducible Poly's:

Formal Derivative:

If $f(x), g(x) \in F[x]$

1. $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
2. $[\alpha f(x)]' = \alpha f'(x)$
3. $[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$

$$f(x) = (x - r_1)^2 (x - r_2)$$

$$f'(x) = 2(x - r_1)(x - r_2) + (x - r_1)^2(1)$$

$$= (x - r_1)\left[2(x - r_2) + 1\right]$$

A field F is <u>perfect</u> iff char $F = 0$ or char $F = p$, prime. and if char $F = p$ the mapping $\underline{F^p = F}$

$x \longmapsto x^p$ Frobenius map

$\mathbb{Z}_5$

1	1
2	2
3	3
4	4
5	0

e is $\mathbb{Q}^c$

$\sum_{n=1}^{\infty} \frac{1}{10^{n!}} = \lambda$

$\mathbb{Q}(\sqrt{2})$

$\mathbb{Q}(\pi)$ $\mathbb{Q}(e)$

2) $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$

$\downarrow$

$$\left\{ \frac{a+b\sqrt{2}+c\sqrt{3}}{a'+b'\sqrt{2}+c'\sqrt{3}} : a, b, c, a', b', c' \in \mathbb{Q} \right\}$$