4/1

Vector Space

Set of elements V and a field $\mathbb{F}$ such that

V forms an abelian group under $\oplus$,

$\mathbb{F}$ operates on elements of V according

to following rules: $\alpha, \beta \in \mathbb{F}$, $u, v \in V$

Scalar multiplication is denoted by "juxta-

$\alpha(u \oplus v) = \alpha u \oplus \alpha v$ dist

$(\alpha + \beta) \cdot u = \alpha u \oplus \beta u$

$\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u$

$1 u = u$

Relax the field to commutative ring,

vector space becomes a "module".

Note: ring is a module over itself

$\mathbb{F}[x]$

$\mathbb{Z}_p[x]$ is vector space over $\mathbb{Z}_p$

Ex: If E is a field & $F \subsetneq E$ is a subfield,

then E is a vector space over F

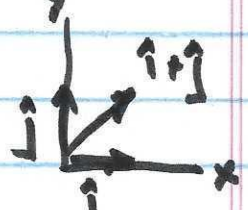
$\mathbb{R}$ is vector space over F

Linear independence.

In $\mathbb{R}^3$, $\hat{i}, \hat{j}, \hat{k}$ are linearly independent

If $\alpha \hat{i} + \beta \hat{j} + \gamma \hat{k} = \hat{0} \Rightarrow \alpha, \beta, \gamma = 0$

$\{\hat{i}, \hat{j}, \hat{k}\}$ is a linearly independent subset of $\mathbb{R}^3$. How about $\{\hat{i}, \hat{j}, \hat{i}+\hat{j}\}$ as lin. indep. subset of $\mathbb{R}^2$.



$\alpha \hat{i} + \beta \hat{j} + \gamma(\hat{i}+\hat{j}) = 0$

$\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$

$\alpha \hat{i} + \beta \hat{j} + \gamma \hat{k} = 0$

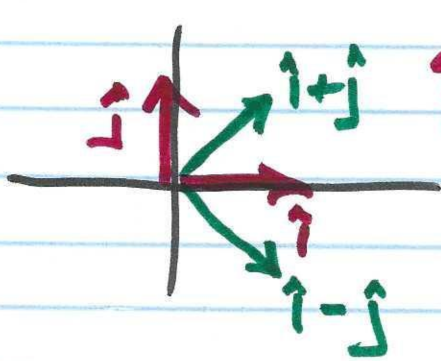
$(\alpha, 0, 0) + (0, \beta, 0) + (0, 0, \gamma) = (0,0,0)$

$\langle V, \mathbb{F} \rangle$

$(\alpha, \beta, \gamma) = (0,0,0)$

$\langle W, \mathbb{K} \rangle$

Among all linearly independent sets of vectors in V there may exist a <u>maximal</u> set. A maximal linearly independent set of vectors is a <u>basis</u>.



$\{\hat{i}, \hat{j}\}$ basis for R^2

$\{\hat{i}+\hat{j}, \hat{i}-\hat{j}\}$

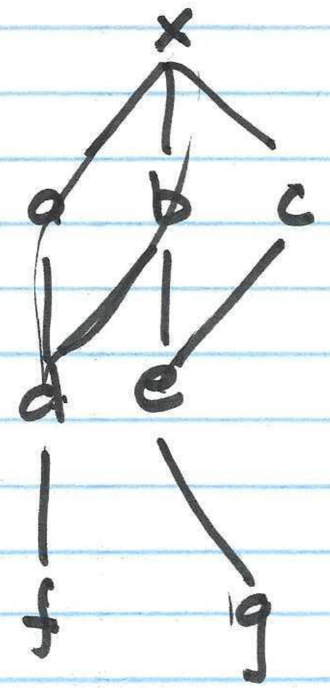
$v_1, v_2 \quad \alpha v_1 + \beta v_2$

$$\mathbb{R}^{\mathbb{N}} = \{(a, b, c, \cdots) : a, b, c \cdots \in \mathbb{R}\}$$

Th^m^ Every vector space has a basis.

Pf: Based on Zorn's Lemma: Given an partially ordered set, S, if every "chain" in S is bounded above <u>in S</u>, then S has maximal elements.

Hasse diagram



Consider all linearly independent sets in V.

There is at least one $\{x\}$.

Suppose $S_1 \subset S_2 \subset S_3 \cdots \subset S_\alpha \subset \cdots$

Look @ $\bigcup_\alpha S_\alpha$. Claim it is lin. indep.

So chain is bounded above by $\bigcup_\alpha S_\alpha$.

Zorn implies <u>maximal</u> element among lin. indep. sets. Can't have element not spanned by $\bigcup_\alpha S_\alpha$, or it could be added w/o destroying lin. indep. — this contradicts maximality. So V has a basis. ∎

Show that the plane whose eqn is

$2x + 3y - z = 0$ is a subspace of $\mathbb{R}^3$.

Use test: closure in both ops:

① Addition $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ $\in$ plane

$$\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}, \quad \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{bmatrix}$$

$$2(x_1 + x_2) + 3(y_1 + y_2) - (z_1 + z_2) \stackrel{?}{=} 0$$

$$\underbrace{(2x_1 + 3y_1 - z_1)}_{0} + \underbrace{(2x_2 + 3y_2 - z_2)}_{0} = 0 \ ?$$

$$2\alpha x_1 + 3\alpha y_1 - \alpha z_1 = \alpha(\underbrace{2x_1 + 3y_1 - z_1}_{0}) = 0$$

THEOREM: EVERY VECTOR SPACE HAS A BASIS

DEFINITIONS:

1) A **vector space** is an ordered pair $(V, \Bbbk)$, where V is an abelian group under the operation **vector addition** and $\Bbbk$ is a field. The field elements act on the group elements via **scalar multiplication**, which applies a scalar to a vector and produces another vector. Since it straddles both the field and group, scalar multiplication must satisfy a number of compatibility properties such as distributivity over vector addition, distributivity over field addition, associativity with field multiplication, and unitarity, i.e. the field unity times any vector returns the vector. We often write the vector space as simply V when the field of scalars is clear from context.

2) If $S \subset V$ is a set of vectors such that every vector in the space can be written as a finite linear combination of elements of S, then S is said to **span** V or be a **spanning set** for V.

3) A set of vectors $I \subset V$ with the property that any finite linear combination of vectors from I that equals the zero vector must have all linear coefficients equal to zero is called a **linearly independent** subset. A spanning set need not be linearly independent (redundancy is allowed). The combination must be finite since an infinite sum of vectors is not defined in the space.

4) A set of vectors $B \subset V$ that is both a spanning set and a linearly independent set is called a **basis** for V. The number of vectors in the basis is the **dimension** of the space. If the dimension of a space is a natural number, the space is a finite dimensional vector space. Vector spaces can have countably infinite or even uncountable bases, in which case they are infinite dimensional.

KEY LEMMA:

Zorn's Lemma, which states that if P is a partially ordered set and every chain (linearly ordered subset) in P has an upper bound in P (this part is essential...the bound has to be inside P), then P has a maximal element relative to its order. There may be more than one maximal element, but at least one. A maximal element has no other element greater or after it in the ordering.

PROOF:

Given the vector space V, consider the collection $\mathfrak{C}$ of all linearly independent subsets of V. This collection is nonvoid, since a subset of one vector is trivially linearly independent. The elements of $\mathfrak{C}$ can be partially ordered by set inclusion. Suppose $S_1 \subset S_2 \subset \cdots \subset S_\alpha \subset \cdots$ is a chain in $\mathfrak{C}$. The notation indicates that the chain could have arbitrary cardinality. Now look at $\bigcup\{S_\alpha \in \mathfrak{C}\}$. This set is linearly independent, since any linear combination in the union must already appear in one of the sets S_α, where it is known to be linearly independent. Therefore, the given chain is bounded above by $\bigcup\{S_\alpha \in \mathfrak{C}\}$. Since this is true of any chain in $\mathfrak{C}$, Zorn's Lemma immediately implies the existence of a maximal element B in $\mathfrak{C}$. Clearly B must span V, otherwise we could adjoin to B any element not in its span, and this would be a linearly independent set larger than B, contrary to its maximality. Hence B is a spanning linearly independent subset of V...a basis. ■