Th^m/ Given ascending chain $I_1 \subset I_2 \cdots I_k \cdots$ in comm ring R.
Show that $\bigcup_k I_k = I$ is an ideal.

Pf: Given $a, b \in I$ need to show $a-b \in I$ and for all $r \in R$, $ra = ar \in I$

Note $a \in I_k$ for some k, and $b \in I_l$ for some l. Let $m = \max\{k, l\}$. Then $a, b \in I_m$. So $a-b \in I_m \subset I$.

Pick $r \in R$. Pick $a \in I$, $a \in I_n$, then $ar = ra \in I_n \subset I$. So ideal test is satisfied and $\bigcup_k I_k$ is an ideal. ■

Th^m Given F as field $\underline{F[x]}$ is a UFD

Use euclidean algorithm on poly of least degree in any ideal.

Euclidean domain : Domain where there exists function (measure) $d : D \to \mathbb{N}_+$

(i) $d(x) \leq d(xy)$ for all $x, y \in D$

(ii) $\exists\, q, r \in D$ such that for $a, b \in D$ where $b \neq 0$ $a = bq + r$ where $d(r) < d(b)$ or $r = 0$

Th^m Given D is an ED, then D is a PID.

Pf: Given D as euc. dom, and $I \subset D$ as a non-zero ideal. Find the element of I with least measure, say a.

Claim: $\langle a \rangle = I$. For any $b \in I$ use euc. alg. to write $b = aq + r$, either $r = 0$ or $d(r) < d(a)$.

Note $d(r) < d(a)$ because $d(a)$ was minimal. So conclude $r=0$, which means $b = aq$ i.e. $I = \langle a \rangle$. Hence D is a PID. ■

Implications:

$$\begin{array}{ccccc} ED & \Rightarrow & PID & \Rightarrow & UFD \\ & \not\Leftarrow & & \not\Leftarrow & \\ & (1949) & & & \end{array}$$

$\mathbb{Z}[x]$ is a UFD - but not PID

$\underline{\langle 2, x \rangle} \subset \mathbb{Z}[x]$.

Thm D is UFD $\Rightarrow$ $D[x]$ is UFD

Counterexample:

$D = \mathbb{Z}[\sqrt{-5}]$ is not a UFD.

$\{a+b\sqrt{-5} : a,b \in \mathbb{Z}\}$

$(a+b\sqrt{-5})(c+d\sqrt{-5}) = 2$

$(ac-5bd)+(ad+bc)\sqrt{-5} = 0$

$N(a+b\sqrt{-5}) = |a^2+5b^2|$

What are units of $\mathbb{Z}[\sqrt{-5}]$

$N(x)=1$ iff x is a unit so units are ± 1.

Consider $46 = 2 \cdot 23$

$(1+3\sqrt{-5})(1-3\sqrt{-5}) = 46$

$2 = xy \quad x,y \in \mathbb{Z}[\sqrt{-5}] \quad x,y$ not unit

$N(2) = N(xy) = N(x)N(y) = 4$

So 2 is irred. over $\mathbb{Z}[\sqrt{-5}]$

(5) $|a^2+5b^2|$

Suppose $23 = xy$, neither x nor y unit

$N(x) = 23$ $\underline{a^2+5b^2 = 23}$

So 23 is irreducible.

Check $1 \pm 3\sqrt{-5} = xy$

$N(x)N(y) = 46 = 2 \cdot 23$

So all four factors are irreducible

and we see there are two distinct

factorizations of 46 in $\mathbb{Z}[\sqrt{-5}]$.

Given D is int-domain & $a, b \in D$

Show a and b are associates iff

$\langle a \rangle = \langle b \rangle$. $a = br = a(sr)$
$b = as$

Product of unit $\cdot$ irred = irred.

$ux = ab \Rightarrow x = a(bu^{-1})$