Thm Given PID D, if $a \in D$ is irreducible, it is prime.

Pf: Suppose $\underline{a \mid bc}$. Want to know $a \mid b$ or $\underline{a \mid c}$.

Look @ $\underline{I} = \{ax + by : x, y \in D\}$

Since D is a PID, $\underline{I = \langle d \rangle}$.

Now $a \in D$, so $a = dr$. Since a is irreducible, either d or r is a unit.

(i) if d is a unit, then $I = D$. If so, we can write $1 = ax + by$. Look @

$c \cdot 1 = \underbrace{\underline{acx} + \underline{bcy}}_{\text{a divides this}}$. Does $a \mid c$

so $a \mid c$

(ii) Suppose r is a unit. Claim $\langle d \rangle = \langle a \rangle$

$b \in I$, $\exists t \in D$ $\ni$. $b = at$, so $a \mid b$.

Hence either $a \mid b$ or $a \mid c$.

So a is prime.

$\mathbb{Z}$, $F[x]$ are PID's ... but

$\mathbb{Z}[x]$ is not a PID.

Consider $I = \{ f(x) \in \mathbb{Z}[x] : f(0) \text{ even} \}$

Claim: I is not representable as

$\langle h(x) \rangle$. If this were true there

would be $f(x)$ & $g(x) \in \mathbb{Z}[x]$ such that

$2 = h(x) f(x)$. Both 2 and $x \in I$.

Look @ degree rule.

$\overset{0}{\deg 2} = \overset{0}{\deg h} + \overset{0}{\deg f} \Rightarrow \deg h = 0$

Conclude $\deg h \Rightarrow h = c$.

So $2 = h(1)f(1)$ so $h(1) = \pm 1$ or ± 2

But $1 \notin \langle h(x) \rangle$, so $h(x) = \pm 2$.

Then $x = (\pm 2)g(x) = \pm 2g(x)$.

Def^{n} Unique Factorization Domain (UFD) is a domain where:

(i) every non-zero non-unit is a product of irreducibles

(ii) the factorization is unique up to order of factors (i.e. $15 = 3 \cdot 5 = 5 \cdot 3$)

Euclidean Domain $\underset{\not\Leftarrow}{\Rightarrow}$ PID $\underset{\not\Leftarrow}{\Rightarrow}$ UFD

Def^n/ ACC: a domain D has the ACC condition (ascending chain condition) iff given a sequence of nested ideals $I_1 \subset I_2 \subset I_3 \cdots \subset I_k \cdots$, is in fact a ~~$\bigcup_k I_k$ is an ideal~~ finite sequence. Domain/ring is called noetherian (nöterian)

Th^m/ If D is a PID, then D has ACC.

Pf: Suppose we have $I_1 \subset I_2 \subset \cdots I_k \cdots$

Note: $\bigcup_k I_k$ is a domain.

$\bigcup_k I_k$ is an ideal. Since $\bigcup_k I_k \subset D$

⑤

Since D is a PID, $\bigcup_k I_k$ is generated by single element d i.e. $\bigcup_k I_k = \langle d \rangle$

d must be in some I_m. But then

$I_m = \langle d \rangle = \bigcup_k I_k$, so

$I_1 \subset I_2 \cdots I_k \cdots =$

$I_1 \subset I_2 \cdots I_m \subset I_m \subset I_m \cdots$

So D has the ACC.