Prob's from Ch 17

2 $F \supset D$
field domain

We have $f(x) \in D[x]$ $f(x)$ irred over F

but reducible over D. $f(x) = \alpha g(x)$

α unit

3 If $f(x) \in \mathbb{Z}[x]$ not primitive, then

$\exists \alpha$ s.t. $f(x) = \alpha g(x)$, use contrapositive

4 $f(x) = a_n x^n + \cdots a_1 x + a_0 \in \mathbb{Z}[x]$.

If $r \in \mathbb{Q}$ and $x - r \mid f(x)$, show $r \in \mathbb{Z}$

Let $r = p/q$ then

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$\times q^n$: $$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n = 0$$

$$p(a_n p^{n-1} + \cdots a_1 p) = -a_0 q^n$$

Rational Test

Given $a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0 = 0 = f(x)$

where $a_n, a_0 \neq 0$

Then if $f(x)$ has a rational root it will

be in the set $\left\{ \frac{\text{divisors of } a_0}{\text{divisors of } a_n} \right\}$

Let root $r = p/q$

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$\star \quad \left[a_n p^n + a_{n-1} p^{n-1} q + \cdots a_1 p q^{n-1}\right] + a_0 q^n = 0$$

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots = -a_0 q^n$$

$$a_n p^n + \left[a_{n-1} p^{n-1} q + a_{n-2} p^{n-2} q^2 + \cdots a_1 p q^{n-1}\right] = -a_0 q$$

$\boxed{q \mid a_n}$ $\boxed{p \mid a_0}$

$1x^5 - 2x^4 + 3x^3 - x^2 + 1 = 0$

$+1 \quad 1 - 2 + 3 - 1 + 1 = 2$

$-1 \quad -1 - 2 - 3 - 1 + 1 \neq 0$

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5 F is field $a \in F$, $a \neq 0$

(a) If $af(x)$ is irred over F, then

$f(x)$ is irred over F

FSOC suppose $f(x) = g(x)h(x)$. Then

$af(x) = ag(x)h(x)$ is reducible over F

$\times$.

(b) If $f(ax)$ is irred over F, then

$f(x)$ is irred over F.

Assume $f(x) = g(x)h(x)$. Then

$f(ax) = g(ax) \cdot h(ax)$ is red.

(c) If $f(x+a)$ is irred over F, then $f(x)$ is irred.

Assume $f(x)$ reducible say as $g(x)h(x)$

then $f(x+a) = \underline{g(x+a)h(x+a)}$

6 Suppose $f(x) \in \mathbb{Z}_p[x]$ and is irred over $\mathbb{Z}_p$. p is prime and $\partial f = n$.

Show $\mathbb{Z}_p[x]/\langle f(x)\rangle$ is field w/ n elements

$\mathbb{Z}_{11}[x]/f(x)$

$\partial f = 5$

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$

Typical element of factor ring is

$g(x) + \langle f(x)\rangle = \boxed{\cancel{f(x)q(x)}} + r(x) + \langle f(x)\rangle$

0

Note: $\partial r = 0$ or $\partial r < n$

So $g(x) = \underline{r(x)} + \langle f(x)\rangle$ $\quad b_i \in \mathbb{Z}_p$

p^n choices for $r(x)$ $\quad r(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$

(9) Show x^3+x^2+x+1 is reducable over $\mathbb{Q}$.

$(-1)+1-1+1=0$ ✓

(10) (a) $x^5+9x^4+12x^2+6$ irred by Eisenstein

(b) x^4+x+1

$x^4+4x^3+6x^2+4x+1+x+1+1$.

mod 2 x^4+x+1

(c) x^4+3x^2+3 irred by Eisenstein

(d) x^5+5x^2+1 reducable (mod 5) root is -1

(e) $\frac{5}{2}x^5+\frac{9}{2}x^4+15x^3+\frac{3}{7}x^2+6x+\frac{3}{14}$

$$\frac{1}{14}\left[\underbrace{35x^5+63x^4+210x^3+6x^2+84x+3}_{\text{irred by Eisenstein}}\right]$$