3/11

Factorization

Given domain D and element $\alpha \in D$
if $\alpha \neq 0$ and α is not a unit, we say
α is irreducible over D if — given
factorization of $\alpha = \beta\gamma$, one of β or γ
is a unit.

Ex. $f(x) = 2x^2 + 4 = 2(x^2+2) = 2(x+i\sqrt{2})(x-i\sqrt{2})$

over $\mathbb{Q}$ irreducible

over $\mathbb{Z}$ ~~ir~~reducible

Ex. $f(x) = 2x^2 + 4$

over $\mathbb{R}$ irreducible

over $\mathbb{C}$ reducible

Ex. $x^2 - 2$ $\quad (x+\sqrt{2})(x-\sqrt{2})$ reducible

over $\mathbb{R}$

over $\mathbb{Q}$ irreducible

Ex. x^2+1

over $\mathbb{Z}_3$ no factorization

over $\mathbb{Z}_5$ $(x+2)(x+3)$ reducible

$F[x]$

Reducibility Test for $\partial 2$ or $\partial 3$

If $f(x)$ has degree 2 or 3, then $f(x)$ is reducible iff $f(x)$ has a zero.

Def^n Given polynomial $f(x) \in \mathbb{Z}[x]$ we say $\gcd(a_n, a_{n-1}, \cdots a_1, a_0) = 1$ $f(x)$ is primitive. Also $\gcd(a_n, a_{n-1}, \cdots a_1, a_0)$ is called the content of $f(x)$.

Gauss' Lemma If $f(x) \in \mathbb{Q}[x]$ and $f(x)$ factors over $\mathbb{Q}$ then it factors over $\mathbb{Z}$.

$$\left(3x-\frac{3}{2}\right)\left(2x+\frac{4}{3}\right) = \underline{6x^2+x-2}$$

$$\left(\frac{6x-3}{2}\right)\left(\frac{6x+4}{3}\right) = (2x-1)(3x+2)$$

Mod p Irreducibility Test

Let p be prime & suppose $f(x) \in \mathbb{Z}[x]$ with degree ≥ 1. Let $\bar{f}(x)$ be the polynomial over $\mathbb{Z}_p$ obtained by reducing all coefficients of $f(x)$ over $\mathbb{Z}_p$. Then if $\bar{f}(x)$ is irreducible over $\mathbb{Z}_p$, ~~then~~ and degree of $\bar{f}(x)$ is still $\deg f(x)$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Pf. So assume $f(x) = g(x)h(x)$

Look @ $\bar{g}(x), \bar{h}(x)$

~~$1 \leq \partial g(x) \leq$~~ $1 \leq \partial\bar{g} \leq \partial g < \partial f$ and

$1 \leq \partial\bar{h} \leq \partial h < \partial f$

$$\bar{f}(x) = \bar{g}(x) \cdot \bar{h}(x) \Rightarrow\Leftarrow$$

<u>Ex.</u> $21x^3 - 3x^2 + 2x + 9 = f(x)$

mod 2 we have $x^3 - x^2 + 1$ does not factor

Conclude $f(x)$ is irreducible over $\mathbb{Q}$

Danger sign: Converse not true

$$f(x) = 21x^3 - 3x^2 + 2x + 8.$$
$$\bar{f}(x) = x^3 - x^2 = x^2(x-1) \text{ factors}$$

Look @ $p = 5$

$$\bar{f}^5(x) = x^3 + 2x^2 + 2x + 3$$

$x=2$ $\quad 8 + 8 + 4 + 3$ NO
$x=3$ $\quad 27 + 18 + 6 + 3$ NO
$x=4$ $\quad 64 + 32 + 4 + 3$ NO

NO roots mod 5 $\Rightarrow$ not reducible

Ex. $f(x) = x^5 + 2x + 4$

Try mod 3 $\quad 0 - \text{no}$

$1 = \text{no}$

$2 = \text{no}$

Assume quad factor $x^2 + ax + b$

a \ b	0	1	2
0	N		
1			
2			

Eisenstein - Schönemann Criterion

$$f(x) = a_n x^n + \cdots a_1 x + a_0 \in \mathbb{Z}[x]$$

If $\exists p$, prime $\ni$ $p \nmid a_n$, $p \mid a_k$ for $0 \leq k \leq p-1$ and $p^2 \nmid a_0$ then $f(x)$ is irreducible over $\mathbb{Q}$.

$$f(x) = \underline{2x^6 - 15x^5 + 6x^3 - 9x^2 + 3x - 21 = 0}$$

Choose $p = 3$, check Eisenstein