Polynomial Rings

$$R[x] = \{a_n x^n + \cdots a_1 x + a_0 : a_i \in R\}$$

$(0,1,3,5,8)$ is alternative rep.

Terminology:

① $a_n x^n + \cdots a_1 x + a_0$ is a poly over R

② $a_k x^k$ is a monomial

③ a_k in above is a coefficient

④ a_n is the leading coefficient, a_k for $k > n$ is by convention 0.

⑤ $a_n x^n$ is leading term

⑥ a_0 is constant term

⑦ a_1 is linear term

⑧ If $a_n = 1$, poly is "monic".

$F[x]$ most often.. Division Algorithm works

↳ field

$$f(x) = g(x)\,q(x) + r(x) \qquad \deg r < \deg g$$

$$\partial r < \partial g$$

(9) degree of poly is highest power of the indeterminate that appears

(10) degree of a constant poly is zero

(11) degree of the zero poly is $-\infty$

(12) If $f(x) = g(x) \cdot h(x)$ for some poly $h(x)$, we say $g(x) \mid f(x)$

3

$\underline{x-a}$

$$
\begin{array}{r}
x^5 - 3x^4 + x^3 + 0x^2 + x + 1 \\
x-2 \enclose{longdiv}{x^6 - 5x^5 + 7x^4 - 2x^3 + x^2 - x - 1} \\
\underline{x^6 - 2x^5} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
-3x^5 + 7x^4 \qquad\qquad\qquad\qquad\qquad \\
\underline{-3x^5 + 6x^4} \qquad\qquad\qquad\qquad\qquad \\
x^4 - 2x^3 \qquad\qquad\qquad\qquad \\
\underline{x^4 - 2x^3} \qquad\qquad\qquad\qquad \\
0 + x^2 \qquad\qquad\qquad \\
\underline{0 + 0} \qquad\qquad\qquad \\
x^2 - x \qquad\quad \\
\underline{x^2 - 2x} \qquad\quad \\
x - 1 \\
\underline{x - 2} \\
\text{Rem } +1
\end{array}
$$

(a)

$$
\begin{array}{r|rrrrrrr}
2 & 1 & -5 & 7 & -2 & 1 & -1 & -1 \\
 & & 2 & -6 & 2 & 0 & 2 & 2 \\
\hline
 & 1 & -3 & 1 & 0 & 1 & 1 & \boxed{1}
\end{array}
$$

$x^5 - 3x^4 + x^3 + x + 1$ R 1

$$2x-3 \,\overline{\big)\, 5x^4 - 2x^3 + 4x^2 - 3x + 6}$$

$$2\left(x - \frac{3}{2}\right)$$

$$\begin{array}{r|rrrrr}
\frac{3}{2} & 5 & -2 & 4 & -3 & 6 \\
 & \downarrow & \frac{15}{2} & \frac{33}{4} & \frac{147}{8} & \frac{369}{16} \\
\hline
\rightarrow & 5 & \frac{11}{2} & \frac{49}{4} & \frac{123}{8} & \boxed{\frac{465}{16}}
\end{array}$$

Rem

$$\frac{1}{2}\left(5x^3 + \frac{11}{2}x^2 + \frac{49}{4}x + \frac{123}{8}\right) + \frac{465}{32}$$

Th^m If D is a domain, then so is $D[x]$.

Pf: Claim $D[x]$ is ring with identity 1

Need to show no zero divisors.

Consider $f(x) = a_n x^n + \cdots a_1 x + a_0$ and

$$g(x) = b_m x^m + \cdots b_1 x + b_0$$

Set $f(x)g(x) = 0$

$$f(x)g(x) = \underline{a_n b_m} x^{n+m} + \cdots a_0 b_0$$

But $a_n b_m$ cannot be zero, since $a_n, b_m \in D$ which admits no zero divisors. ■

Remainder Th^m : If $f(x) \in F[x]$ then if $f(x)$ is divided by linear factor $x-a$, the ~~result~~ remainder is $f(a)$

$$f(x) = (x-a)q(x) + r(x)$$

$$f(a) = \underbrace{(a-a)q(a)}_{0} + r(a)$$

Factor Th^{m}: If $f(x) \in F[x]$, then

$(x-a) \mid f(x) \Leftrightarrow f(a) = 0$ i.e. a is a

zero
<u>root</u> of f <u>root</u> vs <u>zero</u>

$\Rightarrow$ If $(x-a) \mid f(x)$ then $f(a) = 0$

So $f(x) = (x-a)g(x) + 0$

$f(a) = 0 + r(a) = 0$

$\Leftarrow$ If $f(a) = 0$ then $(x-a) \mid f(x)$

then $r(a) = 0$ by Remainder Th^{m}

$f(x) = \underline{(x-a)q(x)} + 0$

so $(x-a) \mid f(x)$

$f(x) = (x-1)^2 \cdot (x-2)$

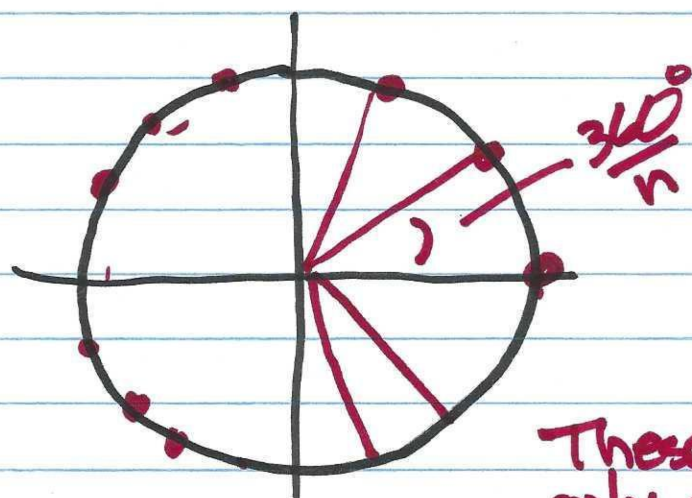
$f'(x) = 2\underline{(x-1)} \cdot (x-2) + \underline{(x-1)}^2(1)$

Show $x^2 + 3x + 2$ has <u>four</u> zeroes over $\mathbb{Z}_6$

$(x+1)(x+2)$

	#1	#2	
$x=2$	3	4	$=0$
$x=1$	2	3	$=0$
$x=4$	5	6	$=0$
$x=5$	0	1	$=0$

<u>Bezout</u> domain



$z^n = 1$

These n roots are only solutions

P.I.D. Principal Ideal Domain

Th^m If F is a field, then $F[x]$ is a PID.

Pf: By earlier theorem, since F is also a domain, then $F[x]$ is a domain.

Let I be an ideal in $F[x]$. If $I = \{0\}$, then $I = \langle 0 \rangle$. So pick $f(x) \in I$. Let $g(x) \in I$ have minimal degree. Want to show $I = \langle g(x) \rangle$. Clearly, $\langle g(x) \rangle \subset I$. So apply $f(x) = g(x)q(x) + r(x)$. Note $\partial r < \partial g$ or $\partial r = 0$. Solving for $r(x) = f(x) - g(x)q(x)$. Since $g(x)$ has minimal degree, it must be the case that $\partial r = 0$, or $\underline{f(x) = g(x) \cdot q(x)}$. But then $f(x) \in \langle g(x) \rangle$, so $F[x]$ is a PID $\blacksquare$

If R is commutative ring, show char $R[x]$ is same as char R.