$\phi : R \to S$

$r \in R$ $A \subset R$ $B \subset S$ ① 2-11

① Show $\phi(nr) = n\phi(r)$ for

① Base case $\phi(r+r) = \phi(r) + \phi(r)$ O.P.

② Ind Hyp $(n-1)\phi(r) = \phi((n-1)r)$

$$\phi(r + (n-1)r) = \phi(r) + (n-1)\phi(r)$$
$$\phi(nr) = \quad = n\phi(r) \blacksquare$$

② $\phi(A) = \{\phi(a) : a \in A\} \subset S$

Let $x, y \in \phi(A)$, then $\exists\, r, s \in A \ni$

$x = \phi(r)$, $y = \phi(s)$

Need $\phi(r) - \phi(s) \in \phi(A)$

$\downarrow$

$\phi(r-s)$ but $r - s \in A$ so

$\phi(r-s) \in \phi(A)$ so subtraction

part of subring test holds.

Note: $r \cdot s \in A$ So $\phi(r \cdot s) \in \phi(A)$ but

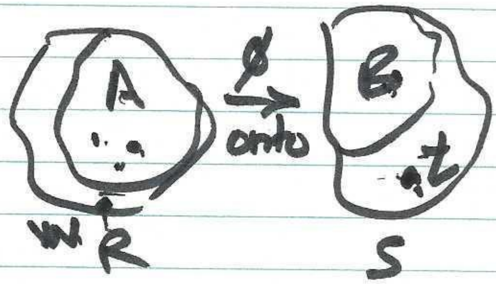
by O.P. $\phi(r \cdot s) = \phi(r) \cdot \phi(s)$ so

$x, y \in \phi(A)$, so is $x \cdot y \in \phi(A)$

3 If $A \subset R$ is ideal, and ϕ is <u>onto</u> S, then $\phi(A)$ is also an ideal.

From #2 $\phi(A) \subset S$ is a subring so 1st half of ideal test is automatic.

Need to show absorption in $\phi(A)$, so pick $x, y \in \phi(A)$ and let $r \in A$, $s \in A$ such that $\phi(r) = x$, $\phi(s) = y$. Note this specification would be invalid if ϕ were not surjective, i.e. if ϕ doesn't have range S, there would be some $z \in S \setminus \phi(R)$.



Let $w = \phi^{-1}(t)$

Given $t \in S$, must show that $tx \in \phi(A)$

Consider $\phi(r \cdot w) = \phi(r) \cdot \phi(w)$

$$\underbrace{\phi(r \cdot w)}_{\in A \text{ by absorption}} = \underbrace{\underbrace{\phi(r)}_{x} \cdot \underbrace{\phi(w)}_{t}}_{\text{back in } \phi(A)}$$

4 $\phi^{-1}(B) = \{r \in R : \phi(r) \in B\}$ is ideal

Pick $\underbrace{r, s}_{\in R} \in \phi^{-1}(B)$. Need $r - s \in \phi^{-1}(B)$

But $\phi(r-s) = \phi(r) - \phi(s) \in B$ by O.P.

$\phi^{-1}[\phi(r) - \phi(s)] = r - s \in \phi^{-1}(B)$

so $\phi^{-1}(B)$ is closed under subtraction.

Now $\phi(r \cdot s) = \phi(r) \cdot \phi(s) \in B$ by O.P.

So pick $t \in R$. Want $r \cdot t \in \phi^{-1}(B)$

~~Apply~~ $\cancel{\phi^{-1}(r \cdot t) = \phi^{-1}(r)\phi^{-1}(t)}$

Consider $\phi(r) \cdot \phi(t) = \phi(rt) \in B$

Now by absorption so $\phi^{-1}(\phi(rt)) \in \phi^{-1}(B)$

So $rt \in \phi^{-1}(B)$.

5 R commutative $\Rightarrow \phi(R)$ commutative

Pick $a, b \in \phi(R)$ Want $ab = ba$

Let $\phi^{-1}(a) = x$ $\phi^{-1}(b) = y$

$\underbrace{xy = yx}_{\text{in } R}$ $\quad \phi(xy) = \phi(x)\phi(y) \leftrightarrow ab$

$\phi(yx) = \phi(y)\phi(x) \leftrightarrow ba$ $\quad =$

(6) Given R with $1_R \in R$ and ~~R~~ $S \neq \{0\}$,
then if ϕ is onto, $\phi(1_R) = 1_S$

$$\boxed{1_R \cdot 1_R = 1_R}$$

$$\phi(1_R \cdot 1_R) = \phi(1_R)\cdot\phi(1_R) = \phi(1_R)$$

can't be zero

$\underline{a^2 = a}$

So $\phi(1_R) = 1_S$

(7) $\phi : R \to S$ $\boxed{\phi^{-1}(0) = \text{Ker}\,\phi}$

So $\text{Ker}\,\phi \subset R$

Assume ϕ is onto.

Suppose $\overset{s_1}{\phi(r_1)} = \overset{s_2}{\phi(r_2)}$

Then $\phi(r_1) - \phi(r_2) = \phi(r_1 - r_2) = 0$

$$\begin{array}{l} r_1 \to \phi(r_1) \\ \neq \\ r_2 \to \phi(r_2) \end{array}$$

are these different