Chapter 12 prob's

1. Finite non-commutative ring?

   $M_k[\mathbb{Z}_n]$ $\underline{k>1\ n>1}$

2. Infinite non-commutative ring w/o identity

   $M_2[2\mathbb{Z}]$ - no identity

3. Find identity in ring $\mathbb{Z}_{10} \supset \{2,4,6,8,0\}$

   6 acts as identity

4. Define: $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Z}\}$

   Consider $(a+b\sqrt{2})(c+d\sqrt{2}) = 2$

   $(ac+2bd)+\sqrt{2}(ad+bc) \in \mathbb{Z}[\sqrt{2}]$

5. Fixed $a,b \in$ ring R the equation $ax=b$ can have more than one sol'n.

   $\mathbb{Z}_4$ $2x=2$ $x=1,3$

6 Show inverses in Ring R are unique.

r^{-1} & $\bar{r}^{-1}$ be inverses of $r \in R$

$\bar{r}^{-1} = \underbrace{r^{-1} \cdot r} \cdot \bar{r}^{-1} = r^{-1}$ so $\underline{\bar{r}^{-1} = r^{-1}}$

9 Let $R_\alpha \subset R \quad \forall \alpha \in \Lambda$

Look @ $\bigcap_{\alpha \in \Lambda} R_\alpha = \bar{R}$

So pick $a, b \in \bar{R}$

(i) Show $a - b \in \bar{R}$

Note $a, b \in R_\alpha \ \forall \alpha \in \Lambda$

So because R_α is ring

then $a - b \in R_\alpha$ ∴ $a - b \in \bar{R}$

(ii) Show $a \cdot b \in \bar{R}$

$a \cdot b \in R_\alpha \Rightarrow a \cdot b \in \bar{R}$

So subring test verifies $\bar{R} \subset R$ is subring

12 Want non-commutative ring w/ 16 elements

$$M_2[\mathbb{Z}_2] \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ & \end{bmatrix}$$

13 Find all subrings of $\mathbb{Z}$

Claim: $k\mathbb{Z} \subset \mathbb{Z}$

$ka - kb = k(a-b) \in k\mathbb{Z}$ ✓

$ka \cdot kb = k^2ab = k(kab) \in k\mathbb{Z}$ ✓

15 Show if $m, n \in \mathbb{Z}$ & $a, b \in R$

then $(m \cdot a)(n \cdot b) = \underline{(m \cdot n)(a \cdot b)}$

$$(\underbrace{a + a + \cdots + a}_{m \text{ terms}})(\underbrace{b + b + \cdots b}_{n})$$

$mn(ab)$

(17) If R is cyclic as a group under $+$, then it is a commutative ring.

Want $a \cdot b = b \cdot a$

Let x generate $\langle R, + \rangle$

Let $a = \underbrace{x + x + \cdots x}_{m} = m \cdot x$

Let $b = \underbrace{x + x \cdots x}_{n} = n \cdot x$

mult in $\mathbb{Z}$

$$a \cdot b = (m \cdot x)(n \cdot x) = (m \cdot n)(x \cdot x)$$

$$b \cdot a = (n \cdot x)(m \cdot x) = n \cdot m\,(x \cdot x)$$

So $mnx^2 = nmx^2$ so $a \cdot b = b \cdot a$ ✓ ■

(19) Center of ring R is $C = \{x \in R \mid ax = xa, \forall a \in R\}$

Show center of R is subring.

If $x, y \in C$ look @ $x - y \overset{?}{\in} C$

Is $(x-y)a = a(x-y)$ for $\forall a \in R$

$$xa - ya = ax - ay \checkmark$$

$x, y \in C$, is $xy \in C$

True if $xya = axy$

$$xya = xay = axy$$

20 Describe elements of $M_2[\mathbb{Z}]$ that have inverses.

$$\det A \cdot \det B = \det(A \otimes B)$$

$$|\det A| = 1 \qquad\qquad \det(A \otimes B) = 1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : |ad - bc| = 1$$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \frac{1}{ad-bc}$$

6

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} =$$

$$\frac{(ad-bc)}{ad-bc}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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23 Find $U(\mathbb{Z}[i])$ $\quad \underline{\pm 1, \pm i}$

$a+bi$

$$\frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$

$\frac{a}{a^2+b^2}$ must be integer

$$c = \frac{a}{a^2+b^2} \quad d = \frac{-b}{a^2+b^2}$$