Ex 15: If $A \overset{\leq}{\subset} R$ and $1 \in A$ then $A = R$.

Note that since A is an ideal, it is absorbing, so consider $\{1 \cdot r : r \in R\}$. This set is contained in A. But it also equals R, so

$A \supset R \supset A$ so $A = R$ ∎

Ex 23: Show that $B = \{br + a : r \in R, a \in A\}$ contains A. Let $r \in R$ be zero, then $\forall a \in A$, $br + a = a$, so $A \subseteq R$,

Maximal Ideal Th^m

Recall M is maximal in R whenever $M \subset R$ and if $M \subset B \subset R$ then either $B = M$ or $B = R$.

~~[illegible] Suppose~~

Th^m: If $M \subset R$ is maximal & R is comm/unital then R/M is field. This is an iff.

(2)

($\Rightarrow$) Suppose R/M is a field. Want to show M is maximal. So let $R \supsetneq B \supsetneq M$

$B \setminus M \neq \emptyset$ So pick $b \in B \setminus M$. Then $b+M \neq$ zero in R/M. By field property of R/M there must be an inverse for $b+M$, say $c+M$. Note $(b+M)(c+M) = 1+M$

Since $b \in B$ and B is an ideal, B is absorbing. Observe $bc \in B$

But $(b+M)(c+M) = \underline{1+M = bc+M}$

So $1-bc \in M \subset B$. So $(1-bc)+(bc)$ is in B, but that means $1 \in B$ and that means (by Ex 15) that $B = R$ $\times$

($\Leftarrow$) Suppose M maximal, want R/M to be field.

$R \setminus M \neq \emptyset$, so let $b \in R \setminus M$. $(b \neq 0)$

Want $b+M$ to have an inverse.

So... consider $B = \{br+m : r \in R, m \in M\}$

Claim: this ideal of R that contains M.

(Proof: Ex 23)

Now M is maximal, so $B = R$! Then $1 \in B$.

Write $1 = bc + \hat{m}$, $\hat{m} \in M$

Then $1 + M = bc + \underbrace{\hat{m} + M} = bc + M$

But $(b+M)(\underline{c+M}) = 1 + M$

So $b+M$ is arbitrary non-zero element of factor ring and has been shown to have inverse $c+M$. Hence R/M is field $\blacksquare$

Need counterexample for prime ideal <u>not</u> being maximal.

$\langle x \rangle$ is prime in $\mathbb{Z}[x]$

Why is $\langle x \rangle$ not maximal.

Note $\overset{\text{no const}}{\langle x \rangle} \subsetneq \overset{\text{even const.}}{\langle 2, x \rangle} \subsetneq \overset{\text{any const.}}{\mathbb{Z}[x]}$.

Ex. $\langle x \rangle = \{ f(x) \in \mathbb{Z}[x] : f(0) = 0 \}$

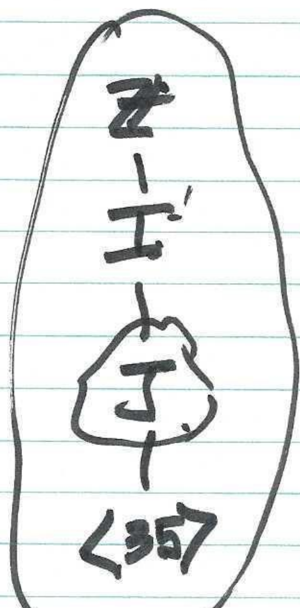
(14) Let $A, B \subset R$, show $\underline{AB} \subset A \cap B$

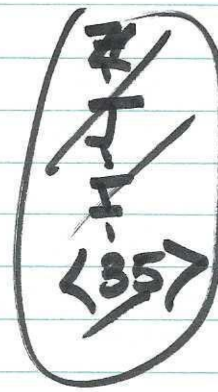
ideals

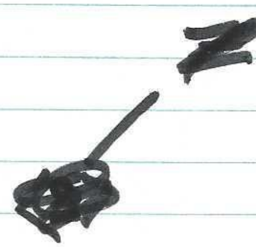
$ab \in A$ A abs

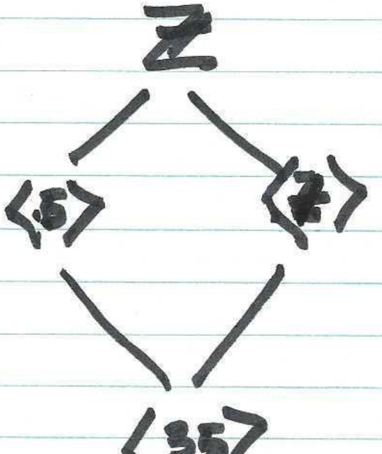
$ab \in B$ B abs.

(18) Suppose in $\mathbb{Z}$ the ideal $\langle 35 \rangle$ is proper ideal of J and $J \subset \mathbb{Z}$. What are the









20 Let $I = \langle 2 \rangle$ Show $I[x]$ is <u>not</u> maximal in $\mathbb{Z}[x]$, yet I is maximal in $\mathbb{Z}$.

$I[x] \subsetneq \langle 2, x \rangle \subsetneq \mathbb{Z}[x]$

$\langle x \rangle$

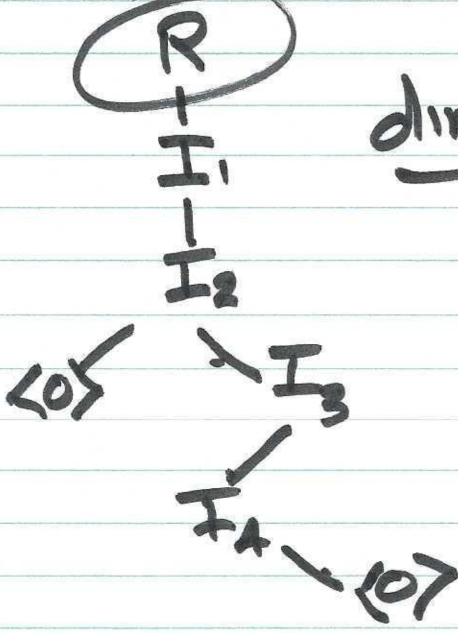
even coeff only & ~~constant~~ constant even

even const all coeff

all coeff & const

25 Only ideals of a field F are $\{0\}$ & F

<u>Dimension to a ring</u> "Krull dimension"



<u>dim 5</u>