Prime Ideals

Given ring R, commutative and unital

If A is a prime ideal, then

R/A is an integral domain & v.v.

Pf: ($\Rightarrow$) Suppose A is prime ideal.
Show R/A is a domain.

R/A is domain, we show A is prime.
Consider $a, b \in R$ & look @ $(a+A)(b+A) =$
$ab + A$. If $(a+A)(b+A) = 0 + A$
$ab + A = 0 + A \Rightarrow ab \in A$
Note that either $a + A = 0 + A$ or
$b + A = 0 + A$, so either $a \in A$ or $b \in A$.

($\Leftarrow$) Note R/A is commutative ring with identity (ex. $(1+A)(x+A) = \underline{1 \cdot x + A}$

Want to show if A is <u>prime</u>, then R/A is a domain.

Must establish if $(a+A)(b+A) = 0+A$, then either $a+A$ or $b+A = 0+A$

$(a+A)(b+A) = ab+A = 0+A$
$\Rightarrow ab \in A$ so by primality $a \in A$ or $b \in A$
But that means either $a+A = 0+A$ or $b+A = 0+A$. So R/A is a domain.

Examples. Let $\mathbb{Z}[x]$ be commutative w/ identity

Consider $\dfrac{\mathbb{Z}[x]}{\langle x \rangle} \cong \mathbb{Z}$

x, x^2, x^3, x^4

Is this prime?

(*) Suppose R/A and $1 \in A \Rightarrow R = A$

Suppose R/A and $u \in A$ (u is unit)

Given $I = \langle 2+2i \rangle \subset \mathbb{Z}[i]$

Show I not prime.

$-2+2i$

$\underline{(1+i)(1-i)} = 2$

$2 \cdot \underline{(1+i)} = 2+2i \in I$

↑

D is called a principal ideal domain (PID) if <u>every</u> ideal is of form $\langle a \rangle$ for some $a \in D$. $\langle a \rangle = \{ad : d \in D\}$

Show $\mathbb{Z}$ are a PID.

But all ideals of $\mathbb{Z}$ are of form $n\mathbb{Z}$

④

Let R be commutative ring & $A \subset R$

Show that the annihilator of A :=

$Ann(A) := \{r \in R : ra = 0, a \in A\}$ is an ideal.

Ideal Test:

Given r_1 & $r_2 \in Ann(A)$, is $r_1 - r_2 \in Ann(A)$?

$(r_1 - r_2)a = r_1 a - r_2 a = 0 - 0 = 0$ ✓

If $r_1 \in Ann(A)$ & $r \in R$, must have ✓

$\boxed{r_1 r = 0.}$ No! $r_1 r a = 0 = r \underbrace{r_1 a}_{0} = r \cdot 0 = 0$

Let R be commutative ring and $A \subset R$ is ideal

$\underline{A \trianglelefteq R}$

Define the "nilradical" of A

$\underline{N(A)} = \{r \in R : r^n \in A\}$ [n depends on r]

Is $N(A)$ an ideal (of R)?

Ideal Test

(i) Choose $r_1, r_2 \in N(A)$. Want to have

$r_1 - r_2 \in N(A)$

$(r_1 - r_2)^{\lambda} \in N(A)$ means

$(r_1 - r_2)^{\lambda} \in A$. Say $r_1^{n} \in A$, $r_2^{m} \in A$

$$(r_1 - r_2)^{\lambda} = \sum_{k=0}^{\lambda} \binom{\lambda}{k} r_1^{k} r_2^{\lambda - k}$$

$\lambda = m + n$

(ii) Show $r_1 \cdot r \in N(A)$ where $r_1 \in N(A)$ &

$r \in R$ $(r_1 r)^{n}$ where $r_1^{n} \in N(A)$

$\underset{N(A)}{\underset{\cap}{r_1^{n}}} \cdot \underline{r^{n}}$

In $\mathbb{Z}_5[x]$, let $\langle x^2 + x + 2 \rangle = I$

Find multiplicative inverse of $\underline{(2x+3) + I}$ in

$$\frac{\mathbb{Z}_5[x]}{I}.$$

$$x^2 = -x-2$$

$$p(x) + \overbrace{\langle x^2 + x + 2 \rangle}^{0} \qquad (2x+3) + \langle x^2+x+2 \rangle$$

$$\left[(ax^2+bx+c) + \langle x^2+x+2 \rangle\right] \cdot \left[(2x+3) + \langle x^2+x+2 \rangle\right]$$

$$= 1 + \langle x^2+x+2 \rangle$$