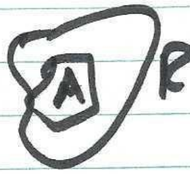
1-21

Ideals



If R is a ring and a subring A, A is called an ideal of R if: A is absorptive.

Absorption means $\forall a \in A$ and $r \in R$ that $ar \in A$ and $ra \in A$.

Ideal Test: Use subring test with $ar, ra \in A$ instead of $ab \in A$ where $a, b \in A$

$G \trianglerighteq H \quad ghg^{-1} \in H \quad \forall h \in H, g \in G$

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1. $\{0\}$; R are ideals of R
2. $n\mathbb{Z}$ ideals of $\mathbb{Z}$
3. R commutative, take $a \in R$, then collect all products

$A = \{ar : a \in R ; r\ R\}$ "a" generates the ideal

Ideals generated by single elements are "principal".

4. Repeat generation construction

with finite set, say $\{a_1, a_2, \cdots, a_n\} \subset R$
then $A = \{a_1 r_1 + a_2 r_2 + \cdots a_n r_n : a_i, r_i \in R\}$ is the ideal generated by the set of a's.

(5) R is $\mathbb{R}[x]$.. claim $A = \underline{\{f(x) \in \mathbb{R}[x] : f(0) = 0\}}$ is an ideal.

$$x^2 + 2x + 1 \in \mathbb{R}[x],$$

$$x^3 - x \in A.$$

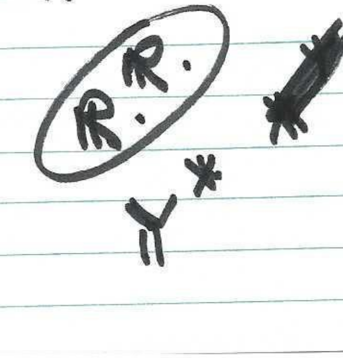
$A = \langle x \rangle$ $\langle \cdot \rangle$ means generated by "·".

$\langle x \rangle \subset \underline{R[x]}$

$\langle x^2 \rangle = ?$

(6) $\mathbb{Z}[x]$, define $I \subset \mathbb{Z}[x]$ where I consists of only polynomials w/ even ~~coefficients~~ constant term.

(7) Let $R = \{f : \mathbb{R} \to \mathbb{R}\}$.



Factor Rings

$$\text{ring} \rightarrow \frac{R}{A} = \{r + A : r \in R\}$$
ideal of R $\rightarrow A$

Define $\underline{(r+A)(s+A)} = rs + A$ (factor ring multiplication)

Example 1 $\frac{\mathbb{Z}}{4\mathbb{Z}} = \{0+4\mathbb{Z}, 1+4\mathbb{Z}, 2+4\mathbb{Z}, 3+4\mathbb{Z}\}$

$4+4\mathbb{Z} = 0+4\mathbb{Z}$ etc.

$$(2+4\mathbb{Z})(3+4\mathbb{Z}) = 6+4\mathbb{Z} =$$

$$2+4+4\mathbb{Z} = 2+4\mathbb{Z}$$

Example 2 $\frac{2\mathbb{Z}}{6\mathbb{Z}} = \{0+6\mathbb{Z}, 2+6\mathbb{Z}, 4+6\mathbb{Z}\}$

$$(4+6\mathbb{Z}) + (2+6\mathbb{Z}) = 6+6\mathbb{Z} = 0+6\mathbb{Z}$$

$$(4+6\mathbb{Z}) \cdot (2+6\mathbb{Z}) = 2+6\mathbb{Z}$$

$\times$	0	2	4
0	0	0	0
2	0	4	2
4	0	2	4

(4)

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_i \in \mathbb{Z} \right\}$$

Define $I \subset R$ to be elements of R but only with <u>even</u> a_i

$$a_i, a_i' \in 2\mathbb{Z} \quad \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} - \begin{bmatrix} a_1' & a_2' \\ a_3' & a_4' \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right.$$

all entries $\in 2\mathbb{Z}$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} =$$

$$\begin{bmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{bmatrix}$$

What is R/I? It is $\left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : x_i \in \{0,1\} \right\}$

$$\begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & 8 \\ 4 & -4 \end{bmatrix}}_{\text{"0" in } I} + \boxed{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$$

$\mathbb{Z}[i] = \{ a+bi : a,b \in \mathbb{Z},\ i = \sqrt{-1} \}$

$\langle 2-i \rangle$ is ideal.

$$\frac{\mathbb{Z}[i]}{\langle 2-i \rangle} = a+bi+\langle 2-i\rangle$$

$$4-2i+\langle 2-i\rangle = \langle 2-i\rangle$$

multiplication in this factor ring is essentially multiplication in $\mathbb{Z}_5$