 1-16

Integral domain problems:

① Find ring R with $a, b \in R$, both $a \div b$ are zero divisors, but $a+b \neq 0$ and $a+b$ is not a zero divisor. Try $a=2$, $b=3$

② Let $a \in R$ w/ 1 and $a^n = 0$ $\exists n \in \mathbb{N}$, a is called a nilpotent element of R.

Show $1-a$ is a unit.

Recall $\frac{1}{1-x} = 1 + x + x^2 + \cdots x^k + \cdots \cancel{\cdot}^{n}$

Set up $\frac{1}{1-a} = 1 + a + a^2 + \cdots + a^{n-1}$ ||

$1 = (1 + a + a^2 + \cdots + a^{n-1}) - (a + a^2 + \cdots + \underbrace{a^n}_{0})$

③ Show the set of nilpotent elements in a commutative ring is a subring.

(i) $\times$ closure $a^n = 0$ $b^m = 0$ $n > m$

$(ab)^x = 0$ Let $x = \cancel{\max}\ \min [m, n]$

$\underbrace{ababab \cdots ab}_{x \text{ pairs}} = \cancel{\underbrace{aaa\cdots}_{x}\underbrace{bbb}_{x}}^{0}$

(ii) a, b nilpotent $\Rightarrow$ $a-b$ nilpotent

$a^n = b^m = 0$

$$\text{Try } (a-b)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} a^k b^{n+m-k}.$$

If $k \geq n$, then $a^k = 0$

If $k < n$, " $b^{n+m-k} = 0$

So, by subring test, nilpotents are subring.

(4) Show 0 is only nilpotent element in domain.

Let $a \neq 0$ If a is nilpotent, then $a^n = 0$ for some $n \geq 2$. But... then $a^k \cdot a^{n-k} = a^n = 0$ and $k \geq 1$ shows zero divisor.

(5) Show 0 & 1 are only <u>idempotents</u> in a domain

Idempotency is property that $a^2 = a$

$a^2 = a \Rightarrow a^2 - a = 0$ or $a(a-1) = 0$

So $a = 0$ or $a = 1$.

6 Find a zero-divisor in $\mathbb{Z}_5[i]$

$a+bi$ where $a,b \in \{0,\dots,4\}$

$(a+bi)(a-bi) = 0 = a^2+b^2$ $\qquad z\bar{z} = |z|^2$

So let $z = 1+2i$ $\quad \bar{z} = 1-2i$ and

$z\bar{z} = 0 \mod 5.$

or $z = 3+4i$ $\quad \bar{z} = 3-4i$ same

7 Find idempotent in $\mathbb{Z}_5[i]$

$$(a+bi)^2 = (a^2-b^2) + i\,(2ab)$$

must be 0

~~So tab~~ $=0$. b must be even.

$(a+2i)^2 = a+2i$ no!

$(a+4i)^2 = a+4i$

$(a^2-4) + i(4a)$ $\Big\}$ $4a = 2 \Rightarrow a = 3$

4

$(a^2 - 16) + i(8a)$

$(a^2 - 1) + i(3a) = a + 4i$ all mod 5

$a^2 - 1 = a \ (5)$

$(3+4i)^2 = (9-16) + i(24)$

$3 + i(4)$

Solution: $3+4i$ is idempotent

$(3+i)^2 = 8 + i6$ mod 5 is $\underline{3+i}$

8. Let R be ring w/ 1

If for $a, b \neq 0$ then $ab \neq 0$, show

~~$ab = ba$~~ If $\boxed{ab = 1}$ then $ba = 1$

$aba = a \Rightarrow a - aba = 0$

$a(1 - ba) = 0 \Rightarrow 1 - ba = 0$ or $\boxed{1 = ba}$

9. Let D be a domain. (contains 1)

Define $P = \{n \cdot 1 : 1 \in D\}$

$(n \cdot 1)(m \cdot 1) = (nm) \cdot 1$ ✓

$(n \cdot 1) - (m \cdot 1) \quad (n - m) 1$

Give an example of an infinite domain D

such that (3) $\text{char}(D) = 3$. $\mathbb{Z}_3[x]$

$(x+y)^p = x^p + y^p$

$x, y \in R$ and $\text{char}(R) = p \neq 0$

$$(x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^k y^{p-k}.$$

$$(x+y)^3 = x^3 + \underline{3x^2y + 3xy^2} + y^3$$

If $1 \leq k \leq p-1$

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$