Zermelo-Fraenkel-Choice (ZFC) Axioms

Axioms are basic assertions of what we feel is true about a subject that admits a logical development. We can do no better than note their obviousness, accept them on intuitive grounds, then build a theory on top of them. They introduce primitives (undefined terms) and declare that the primitives have certain properties. You have probably heard the same kind of explanation (apologia) concerning the axioms for plane geometry. Euclid tied himself in knots trying to define primitives ("a point is that which has no extent") but his axiomatic development of geometry has been a model for mathematical theory for two millenia. Sufficiently strong axiom systems (strong enough to support the development of arithmetic) suffer from an unavoidable defect discovered by Kurt Gödel: there can be true statements expressed in the language of the axioms that cannot be affirmed or denied by arguments based on the axioms. Since it had baffled mathematicians for so long, Fermat’s Last Theorem was briefly suspected of being an example of this phenomenon, however it has now been proved with the normal proof machinery. So far, the mathematical community has not detected any contradictions in the set of (slightly redundant) axioms known as the Zermelo-Fraenkel-Choice axiom sytem (ZFC).

The usual specification for an axiom system is that it must be powerful enough to support the theory for which it is intended, yet economical in the number and complexity of the axioms themselves. Any statement that cannot be proved (or disproved) using a given collection of axioms can be appended to them as a new axiom to expand the scope of their theory. The Axiom of Choice, which is "obviously true", was adjoined to the original Zermelo-Fraenkel axioms in this manner. Another very much less obvious candidate to be adjoined to ZFC is the Generalized Continuum Hypothesis, which seems to be independent of ZFC (neither it nor its negation contradict ZFC). Determining this independence drove the development of set theory as its principal problem for many years.

Below are the ZFC axioms, both in words and in logical symbols, along with comments that may make the point of the axioms more memorable. The specific wording of the axioms varies from author to author, and since there is a little redundancy, even the list of axioms varies as well, so don’t be put off if you check another reference and find some differences. Recall that fundamentally all the axioms are declarations about the undefined (but intuitively understood) property of membership or belonging expressed in terms of first order predicate logic. Note that all but one axiom deal with the creation of a set. A brief summary of logical symbols follows:

LOGICAL SYMBOLS

$\wedge$ means "and"

$\vee$ means "or"

$\neg$ means "not"

$\Rightarrow$ means "implies"

$\Leftrightarrow$ means "if and only if"

$\exists$ means "there exists"

$\forall$ means "for every"

! means "unique"

1) **Axiom of Existence**

There exists a set with no elements.

$\exists S(S = \varnothing)$

This axiom assures that we have something to work with...not much, but enough.

2) **Axiom of Extensionality**

If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

$\forall X \forall Y([z \in X \Leftrightarrow z \in Y] \Leftrightarrow X = Y)$

This axiom defines equality of sets.

3) **Axiom Schema of Comprehension**

Given $P(x)$ as a property of x, for any X there is a Y such that $x \in Y$ if and only if $x \in X$ and $P(x)$ is true.

$\forall X \exists Y(Y = \{x : x \in X \wedge P(x)\})$

An axiom (or theorem) schema is a template that depends on a parameter, in this case the property P. There are infinitely many axioms that fit this template, one for each P. In the case of a theorem schema, we would present a proof of the template which would cover all substitution instances of particular properties. This axiom template allows you to collect into one subset all the elements of a given set that satisfy some property or condition.

4) **Axiom of Pair**

For any X and Y, there is a Z such that $z \in Z$ if and only if $z = X$ or $z = Y$.

$\forall X \forall Y \exists Z(z \in Z \Leftrightarrow z = X \vee z = Y)$

This axiom allows two sets to be placed together into another set which is an unordered pairing. We denote that set as $\{X, Y\}$. If $X = Y$, $\{X, Y\} = \{X, X\} = \{X\}$, and this gives us a way of creating nonempty sets. For example, $\{\varnothing, \varnothing\} = \{\varnothing\} \neq \varnothing$.

5) **Axiom of Union**

For any S, there is a U such that $x \in U$ if and only if $x \in X$ for some $X \in S$.

$\forall S \exists U(x \in U \Leftrightarrow x \in X \in S)$

This axiom says that if you have a collection of sets, it is possible to place *exactly* all of the elements that appear in any member of the collection into one big set...the union. This is conventionally written $\bigcup S$. Any redundant appearances of elements in the union are assumed to be eliminated (unlike a multiset). Given A and B, the axiom of pair allows the construction of $S = \{A, B\}$, then $\bigcup S$ becomes the more familiar $A \cup B$.

6) **Axiom of Power Set**

For any S, there is P such that $X \in P$ if and only if $X \subseteq S$.

$\forall S \exists P(X \in P \Leftrightarrow X \subseteq S)$

Here the notation $X \subseteq S$ is shorthand for $x \in X \Rightarrow x \in S$. This axiom says that you can put all the subsets (including $\varnothing$, which is a subset of every set) of a given set into one collection, the power set of the given set. The power set of X is often written $\wp(X)$ or 2^X. The rationale for the second notation is that every subset Y of X corresponds to a possible characteristic function $\chi_Y : X \to \{0, 1\}$ on X, and the usual notation for all functions from A to B is B^A.

7) **Axiom of Infinity**

There is an inductive set.

$\exists I(\varnothing \in I \wedge [x \in I \Rightarrow \{x \cup \{x\} \in I])$

For an element x, its *successor* is defined to be $x \cup \{x\}$. An inductive set has zero (which we will identify with $\varnothing$) in it and the successor of any element, so $\varnothing \cup \{\varnothing\} = 1$, and so forth are also present in every inductive set. If an inductive set has an element α which is not one of the eventual successors of zero, then it must also have all of the eventual successors of α as well. The set that consists purely of $0, 1, \cdots$ will be identified as the set of natural numbers. This axiom introduces the idea of a completed infinity, which some mathematicians still object to, but the axiom of infinity is usually regarded as having the same credibility as the other axioms.

8) **Axiom Schema of Replacement**

Given $P(x, y)$ as a property of x and y, such that for a given x there is a unique y making $P(x, y)$ true, for any X there is a Y such that for every $x \in X$ there is a $y \in Y$ for which $P(x, y)$ is true.

$\forall X \exists Y([x \in X \Rightarrow \exists ! y(P(x, y))] \Rightarrow [Y = \{y : x \in X \wedge P(x, y)\}])$

This axiom seems redundant in light of the comprehension schema. In fact, the comprehension schema can be derived as a theorem from the replacement schema and the other axioms. The point of carrrying redundant axioms is for clarity and convenience, even though the axiom system might not be the most economical The comprehension schema says that all the elements in a set with a certain property can be aggregated into a subset of the given set. The replacement schema says that if you have a given set and each element in it has a unique "correspondent" element in some other (possibly the same) set, then the correspondent elements can be aggregated together into their own set. The correspondent elements are determined by checking if the property $P(x, y)$ is satisfied. In short, comprehension identifies particular elements within a given set; replacement identifies elements in another set connected to the given set by some restrictive property. So why the name "Replacement"? You can view the construction of the set of correspondent elements as a systematic replacement of the elements in the original set.

9) **Axiom of Choice**

Given a collection of pairwise disjoint and nonempty sets, there exists a set consisting of one element from each set in the collection.

$$\forall X\Big[\Big(\forall x \in X\ \forall y \in X(x = y \Leftrightarrow x \cap y \neq \varnothing\Big) \Rightarrow \exists z\Big(\forall x \in X\ \exists! y\ (y \in x \cap z)\Big)\Big]$$

The axiom of choice is sometimes called the multiplicative axiom in connection with cartesian products. It states that there is a "choice function" that allows one to select one element each from the members of a family of nonempty sets. It seems rather trivial for finite families, but the situation is different for infinite families. Invoking a choice function for an infinite family requires adding an axiom specifically for that purpose.

This is the usual list of ZFC axioms given by many textbooks. There are other axioms that are sometimes added which are redundant, such as the Axiom of Intersection and the Axiom of Foundation (aka the Axiom of Regularity). And there are additional candidate axioms that really do add something new, such as the affirmation of the Continuum Hypothesis, the Generalized Continuum Hypothesis, and the Axiom of Inaccessible Cardinals. These candidates are remarkably different from the ZFC axioms in terms of how remote they are from our intuition. Remember...we like our axioms to be intuitively plausible. Nevertheless, it seems that if mathematics is to advance, we will need to unlock further secrets of set theory and incorporate them into the existing axiomatic framework.

Peano Axioms for Arithmetic

Primitives: zero 0, the unary operation S, and the binary operations $+$ and $\cdot$

P1: If $S(n) = S(m)$ then $n = m$
P2: $S(n) \neq 0$
P3: $n + 0 = n$
P4: $n + S(m) = S(n + m)$
P5: $n \cdot 0 = 0$
P6: $n \cdot S(m) = (n \cdot m) + n$
P7: $n \neq 0 \Rightarrow n = S(k)\ \exists k$
P8: **Induction Schema**: Let ϕ be an arithmetical property (expressible in term of the primitives). If 0 has the property ϕ and if $P(k) \Rightarrow P(S(k))$ for every k, then every number has property ϕ