INTRODUCTION:

Any axiomatic system (see the list of ZFC axioms) must start with undefined primitive terms. You can argue that we should be able to define them, but when you try to do it, you find that you are only using other terms that have to be left undefined, hence primitive, in order to avoid circularity. We also assume that we have available all the machinery of first order predicate logic, so quantifiers and the usual truth-functional connectives (and, or, not, implies, etc.) are at hand. Our list of primitives includes: set, element, membership, and property. We usually (but not always) denote a set by a capital letter, but if it is an element of another set, we often use lower case letters. Membership, or belonging, is a property of elements relative to sets, and we denote it by $\in$, which is a stylized epsilon. Properties are denoted by capital letters with parentheses after them indicating which elements are supposed to have those properties, so $P(x,y)$ means x and y have property P.

DEFINITIONS:

Definition 1: The unique set given by the Axiom of Existence is called the **null** (**empty**, **void**) **set**. It is customarily denoted by $\varnothing$.

Definition 2: The unique set given by the Axiom Schema of Comprehension is called **the set of all** $x \in A$ **with property** P. It is denoted by $\{x \in A | P(x)$ is true$\}$ or simply $\{x \in A | P(x)\}$.

Definition 3: The unique set given by the Axiom of Pair is the **unordered pair** $\{A,B\}$. In the event $A = B$, $\{A,A\} = \{A\}$.

Definition 4: The unique set given by the Axiom of Union is called **the union of** S. It is denoted by $\bigcup S$. We call S **a system** (**family**, **collection**) of sets.

Definition 5: $\bigcup\{A,B\} = A \cup B$

Definition 6: A is a **subset of** B if and only if every element of A belongs to B. We denote this by writing $A \subseteq B$.

Definition 7: The unique set given by the Axiom of Power set is called the **power set of** S and consists of all subsets of S. It is denoted by $\wp(S)$ or sometimes by 2^S.

Notation: If it is known that all x with property P belong to some set S, then we write $\{x | P(x)\}$ and call it the set of all x with property P.

Definition 8: The **intersection** of two sets, written $A \cap B$, is the collection of elements common to both sets. $A \cap B = \{x \in A | x \in B\}$.

Definition 9: The **difference** (**relative complement**) of two sets, written $A - B$, is the collection of elements in A but not B. $A - B = \{x \in A | x \notin B\}$. If all set operations are done on the subsets of some given set, often called the **universe** $\mathbb{V}$ (**of discourse**), we write $\mathbb{V} - B$ as B^c. We also have $A - B = A \cap B^c$.

Definition 10: The **symmetric difference** of two sets, written $A \triangle B$, is the collection of elements in either set but not both. $A \triangle B = (A - B) \cup (B - A)$.

Definition 11: **Commutativity** of an operation is the property of being order independent. $a * b = b * a$.

Definition 12: **Associativity** of an operation is the property of being grouping independent. $a * (b * c) = (a * b) * c$

Definition 13: **Distributivity** of an operation $*$ over an operation $\circ$ is the property $a * (b \circ c) = (a * b) \circ (a * c)$

Definition 14: Two sets are **disjoint** if they have empty intersection. A and B are disjoint if $A \cap B = \varnothing$.

Definition 15: A system S of **mutually disjoint** sets has the property that $A, B \in S$ implies $A \cap B = \varnothing$.

Definition 16: The **ordered pair** $(a,b) = \{\{a\},\{a,b\}\}$. The **first entry** (**component**) is a and the **second entry** is b. An **ordered triple** $(a,b,c) = ((a,b),c)$. An **ordered** n**-tuple** $(a_1,\cdots,a_n) = ((a_1,\cdots,a_{n-1}),a_n)$. Note $((a,b),c) \neq (a,(b,c))$.

Definition 17: A set is a **binary relation** if all its elements are ordered pairs.

Definition 18: The **domain** of a relation is the set of all first entries of its ordered pairs, and the **range** is the set of all second entries. The domain of R is $domR$ and the range $ranR$.

Definition 19: The **field** of a binary relation is the union of its domain and range. So $fldR = domR \cup ranR$.

Definition 20: A **binary relation on a set** is one where its field is a subset of the given set.

Definition 21: The **image** of a subset A of the domain of a relation R is the set of all second elements of pairs where an element of A is the first element. $R(A) = \{b \in ranR | a \in domR\}$.

Definition 22: The **inverse image** (**or preimage**) of a subset B of the range of a relation is the set of all first elements of pairs where an element of B is the second element. $R^{-1}(B) = \{a \in domR | b \in ranR\}$. The **inverse relation** (to R) is $R^* = \{(b,a) | (a,b) \in R\}$. It is a lemma that $R^*(B) = R^{-1}(B)$.

Definition 23: The **cartesian product** of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs with first element from A and second element from B. We have $A \times B = \{(a,b) | a \in A, b \in B\}$. A **relation from** A **to** B is a subset of $A \times B$. A **relation on A** is a subset of $A \times A = A^2$. $A^n = A^{n-1} \times A$. The elements of A^n are called n**-tuples**.

Notation: aRb or $a \sim b$ are used to indicate $(a,b) \in R$.

Definition 24: The **composition** of binary relations R and S, written $R \circ S$, is a transitive construction where S is applied first followed by R. We have $R \circ S = \{(x,z) | (x,y) \in S \wedge (y,z) \in R\}$.

Definition 25: A **function** (**mapping**) is a binary relation where two ordered pairs must be equal if their first elements are equal. A function from A to B, written $f : A \to B$, is a subset of $A \times B$ with the property that if $(a,b),(a,c) \in f$, then $b = c$.

Definition 26: A **unary operation** on a set S is a function from a subset of S into S. A **binary operation** on a set S is a function from a subset of S^2 into S. A **ternary operation** on a set S is a function from a subset of S^3 into S. The **arity** of an operation on S is n, where the operation is a function from a subset of S^n into S.

Definition 27: A **structure** is a set consisting of a ground set of elements together with a (finite) collection of relations and operations on the ground set.

Definition 28: An **injective** (**invertible**, **1**:**1**) function $f : A \to B$ has the property that if $(a,c),(b,c) \in f$, then $a = b$. Every relation induces an inverse relation (switch the elements in the ordered pair), but not every function induces an inverse function by this construction because the property in Definition 25 must be preserved. The ones that do are precisely the injective functions.

Notation: If f is invertible as a function we write f^{-1}*. If it is* ***only*** *invertible as a relation and not as a function, we still write* $f^{-1}(B)$ *to mean the inverse image of* $B \subseteq ranf$*. In this situation, where we view f and* f^{-1} *as operating on sets instead of particular elements, they are both functions. Note that as a set function,* f^{-1} *is generally not the inverse of f. See "saturated" below. Some texts make this difference in viewpoint explicit and call* $\widetilde{f}$ the **associated** (with f) **set function**.

Definition 29: A **surjective** (**onto**) function $f : A \to B$ has the property that $ranf = f(A) = B$.

Notation: It is common to call B *the codomain of f, and then surjectivity becomes the statement "the range of f is its codomain".*

Definition 30: A **bijective** function is one that is both injective and surjective.

Definition 31: A family of functions is **compatible** if they agree on their common domain (intersection of all domains).

Definition 32: An **isomorphism** of structures $\{S, R_1, \cdots, R_m, O_1, \cdots, O_n\}$ and $\{S', R'_1, \cdots, R'_m, O'_1, \cdots, O'_n\}$ is a bijective mapping $\phi : S \to S'$ such that for all k, $(a,b) \in R_k$ if and only if $(\phi(a), \phi(b)) \in R'_k$ and $(a, \cdots) \in O_k$ if and only if $(\phi(a), \cdots) \in O'_k$.

Definition 33: A set $A \subseteq domf$ is **saturated** if $f^{-1} \circ f(A) = A$. If it is necessary to emphasize the function, we say the set is f-saturated.

Definition 34: The sets A and B are **equipotent** (**equipollent**, **have the same cardinality**) if there exists a bijective $f : A \to B$.

Notation: This situation will be denoted by $|A| = |B|$. (Here we are only talking about equal cardinality...we have not defined it yet.)

Definition 35: Given a set A, its **successor** is $A \cup \{A\}$. We indicate this also by saying $S(A) = A \cup \{A\}$.

Definition 36: A set I is called **inductive** if $\varnothing \in I$ and for every $A \in I$ we have $S(A) \in I$.

Notation: We label $\varnothing$ as "0", $S(\varnothing) = S(0)$ as "1", and in general $S(n) = n + 1$. The plus sign is not yet defined...this is our naive understanding of "next one".

Definition 37: The set of **natural numbers** $\mathbb{N} = \{x | x \in I \text{ for every inductive set } I\}$. The Axiom of Infinity gives us at least one inductive set, so this definition is not vacuous. Note that zero is a natural number.

Definition 38: The "**less than**" **relation** $m < n$ on the natural numbers is determined by the condition $m < n$ if and only if $m \in n$.

Definition 39: **Transitivity** is a property of a binary relation R on a set A where $(a,b) \wedge (b,c) \in R \Rightarrow (a,c) \in R$.

Definition 40: **Reflexivity** is a property of a binary relation R on a set A where $(a,a) \in R$ for all $a \in A$.

Definition 41: **Symmetry** is a property of a binary relation R on a set A where $(a,b) \in R \Rightarrow (b,a) \in R$.

Definition 42: **Antisymmetry** is a property of a binary relation R on a set A where

$(a,b) \wedge (b,a) \in R \Rightarrow a = b$.

Definition 43: **Asymmetry** is a property of a binary relation R on a set A where $(a,b) \in R \Rightarrow (b,a) \notin R$.

Definition 44: **Comparability** is a property of a binary relation R on a set A where $(a,b) \vee (b,a) \in R$ for all $a,b \in A$.

Definition 45: An **equivalence relation** is a binary relation on a set A with reflexivity, symmetry, and transitivity.

Definition 46: A **partial order** is a binary relation on a set A with reflexivity, antisymmetry, and transitivity.

Notation: If R is a partial order and $(a,b) \in R$, we usually write $a \leq b$ instead of aRb.

Definition 47: A **strict partial order** is a binary relation on a set A with asymmetry and transitivity.

Definition 48: A **linear** (**total**, **complete**) **order** is a partial order with comparability.

Definition 49: A partial order R on the set A has the **common successor property** if whenever $a,b \in A$, there exists a $c \in A$ such that $(a,c) \wedge (b,c) \in R$.

Definition 50: An element $m \in A$ is **minimal** with respect to the partial order R on A if $(a,m) \in R \Rightarrow m = a$.

Definition 51: An element $M \in A$ is **maximal** with respect to the partial order R on A if $(M,a) \in R \Rightarrow M = a$.

Definition 52: An element $m \in A$ is a **minimum** (**least**) with respect to the partial order R on A if $(m,a) \in R$ for all $a \in A - \{m\}$.

Definition 53: An element $M \in A$ is a **maximum** (**greatest**) with respect to the partial order R on A if $(a,M) \in R$ for all $a \in A - \{M\}$.

Comment: Maximal and minimal elements need not exist. If they do exist, they may not be unique. A maximal element need not be a maximum, but a maximum element is surely maximal. Likewise, mutatis mutandis, for minimal elements. Maximum and minimum elements, if they exist, are unique.

Definition 54: A **well**-**ordering** on the set A is a partial order R where every nonempty subset of A has a minimum element with respect to R. A is called a **well**-**ordered set**.

Definition 55: A **sequence** in a set A is a function $f : \mathbb{N} \to A$.

Notation: We denote the generic image $f(n)$ *of* $n \in \mathbb{N}$ *as* a_n *and the totality of the sequence by* $\langle a_n \rangle$.

Definition 56: If $g : A \times \mathbb{N} \to A$ is a function and $a_0 \in A$, we say the sequence $\langle a_n \rangle$ is **generated recursively** (or by **recursion**) if $a_{n+1} = g(a_n, n)$

Definition 57: There exist sets called **cardinal numbers**, and for any given set X, a unique cardinal number c can be placed in bijective correspondence with X. We say the **cardinality** of X is c and write $|X| = c$.

Definition 58: If a set can be placed in bijective correspondence with a natural number $n \in \mathbb{N}$, we say it is **finite** and has cardinality n. If this cannot be done, then the set is **infinite**.

Definition 59: A set X is **countable** (**denumerable**) if there is an injection $f : X \to \mathbb{N}$. This situation is written $|X| \leq |\mathbb{N}|$.

Notation: The cardinal number of $\mathbb{N}$ is $\aleph_0$, read "aleph nought". Aleph is the first letter of the Hebrew alphabet.

Definition 60: A set X is **countably infinite** if there is a bijection $f : X \to \mathbb{N}$. So $|X| = |\mathbb{N}| = \aleph_0$.

Definition 61: A set is **uncountable** if it is not countable.

Definition 62: **Cardinal addition** - if A and B are disjoint sets, then $|A| + |B| = |A \cup B|$

Definition 63: **Cardinal multiplication** - if A and B are disjoint sets, then $|A| \cdot |B| = |A \times B|$

Definition 64: **Cardinal exponentiation** - if A and B are disjoint sets, then $|A|^{|B|} = |\{f | f : B \to A\}|$

Definition 65: A **poset** is a partially ordered set.

Definition 66: A **toset** is a totally ordered set.

Definition 67: A **woset** is a well-ordered set.

Definition 68: An **order isomorphism** is a bijection between two posets that preserves order structure.

Definition 69: Two tosets (or wosets) have the same **order type** if they are order-isomorphic.

Definition 70: The **initial segment** of a woset W determined by $a \in W$ is the set $W[a] = \{w \in W | w < a\}$

Definition 71: A set is **transitive** if each of its elements is also a subset of it.

Definition 72: A set is an **ordinal number** if it is transitive and well-ordered by the membership relation $\in$.

Definition 73: The **order type of a woset** is the unique ordinal number isomorphic to it.

Definition 74: An ordinal is **regular** if it has an immediate predecessor (it is the successor of some ordinal).

Definition 75: An ordinal is a **limit ordinal** if has no immediate predecessor.

Definition 76: **Ordinal addition** - for any ordinal β (i) $\beta + 0 = \beta$, (ii) $\beta + (\alpha + 1) = (\beta + \alpha) + 1$ for any ordinal α, and (iii) $\beta + \alpha = \sup\{\beta + \gamma | \gamma < \alpha\}$ for all limit ordinals $\alpha \neq 0$

Definition 77: **Ordinal multiplication** - for any ordinal β (i) $\beta \cdot 0 = 0$, (ii) $\beta \cdot (\alpha + 1) = (\beta \cdot \alpha) + \beta$ for any ordinal α, and (iii) $\beta \cdot \alpha = \sup\{\beta \cdot \gamma | \gamma < \alpha\}$ for all limit ordinals $\alpha \neq 0$

Definition 78: **Ordinal exponentiation** - for any ordinal β (i) $\beta^0 = 1$, (ii) $\beta^{\alpha+1} = \beta^\alpha \cdot \beta$ for any ordinal α, and (iii) $\beta^\alpha = \sup\{\beta^\gamma | \gamma < \alpha\}$ for all limit ordinals $\alpha \neq 0$

Definition 79: An ordinal α is an **initial ordinal** if it is not equipollent to any $\beta < \alpha$

Definition 80 (see #57): The **cardinal number** of a set X is the unique initial ordinal number equipollent to X.

Definition 81: The **Hartogs number** of the set A is the least ordinal number that is not equipotent to any subset of A.

Definition 82: An infinite cardinal $\aleph_\alpha$ is a **successor cardinal** if its index α is a successor ordinal.

Definition 83: An infinite cardinal $\aleph_\alpha$ is a **limit cardinal** if its index α is a limit ordinal.

Definition 84: If $\langle \alpha_\nu | \nu < \theta \rangle$ is an increasing sequence of ordinals, then $\lim_{\nu \to \theta} \alpha_\nu = \sup\{\alpha_\nu | \nu < \theta\}$

Definition 85: An infinite cardinal is called **singular** if there exists an increasing transfinite sequence $\langle \alpha_\nu | \nu < \theta \rangle$ of ordinals $\alpha_\nu < \kappa$ whose length θ is a limit ordinal less than κ and $\kappa = \lim_{\nu \to \theta} \alpha_\nu$ (for example, $\aleph_\omega$ is a singular cardinal)

Definition 86: A cardinal is **regular** if it is not singular.

Definition 87: An uncountable cardinal that is both a limit cardinal and regular is called **weakly inaccessible**.

Definition 88: If α is a limit ordinal, the **cofinality of** α, denoted $cf(\alpha)$ is the least ordinal number θ such that α is the limit of an increasing sequence of ordinals of length θ.