$|A| = |B|$ iff $\exists \phi$, ϕ bijection

$\phi : A \to B$

Cardinality (equality of) is equiv. rel'n.

① If $\phi : A \to A$ is identity then

ϕ is bijection

② If $|A| = |B| \Rightarrow \exists \phi : A \to B$ s.t.

ϕ is bijection, note however

that $\phi^{-1}: B \to A$ is bijection

hence $|B| = |A|$

③ If $|A| = |B|$ & $|B| = |C|$ then

$\exists \phi, \psi$ s.t. $\phi : A \to B$ bijectively

and $\psi : B \to C$ bijectively

so $\psi \circ \phi : A \to C$ bijectively

$\therefore |A| = |C|$

If $\phi : A \to B$ is injective, then

$\phi : A \to \phi(A)\ [\subset B]$ is bijective

$\Rightarrow |A| = |\phi(A)| \leq |B|$

We say $|A| \leq |B|$ if $\exists\, \phi : A \to B$, inj.

(i) $|A| \leq |A|$ Take ϕ as identity (inj)

(ii) $|A| \leq |B|$ & $|B| \leq |C|$

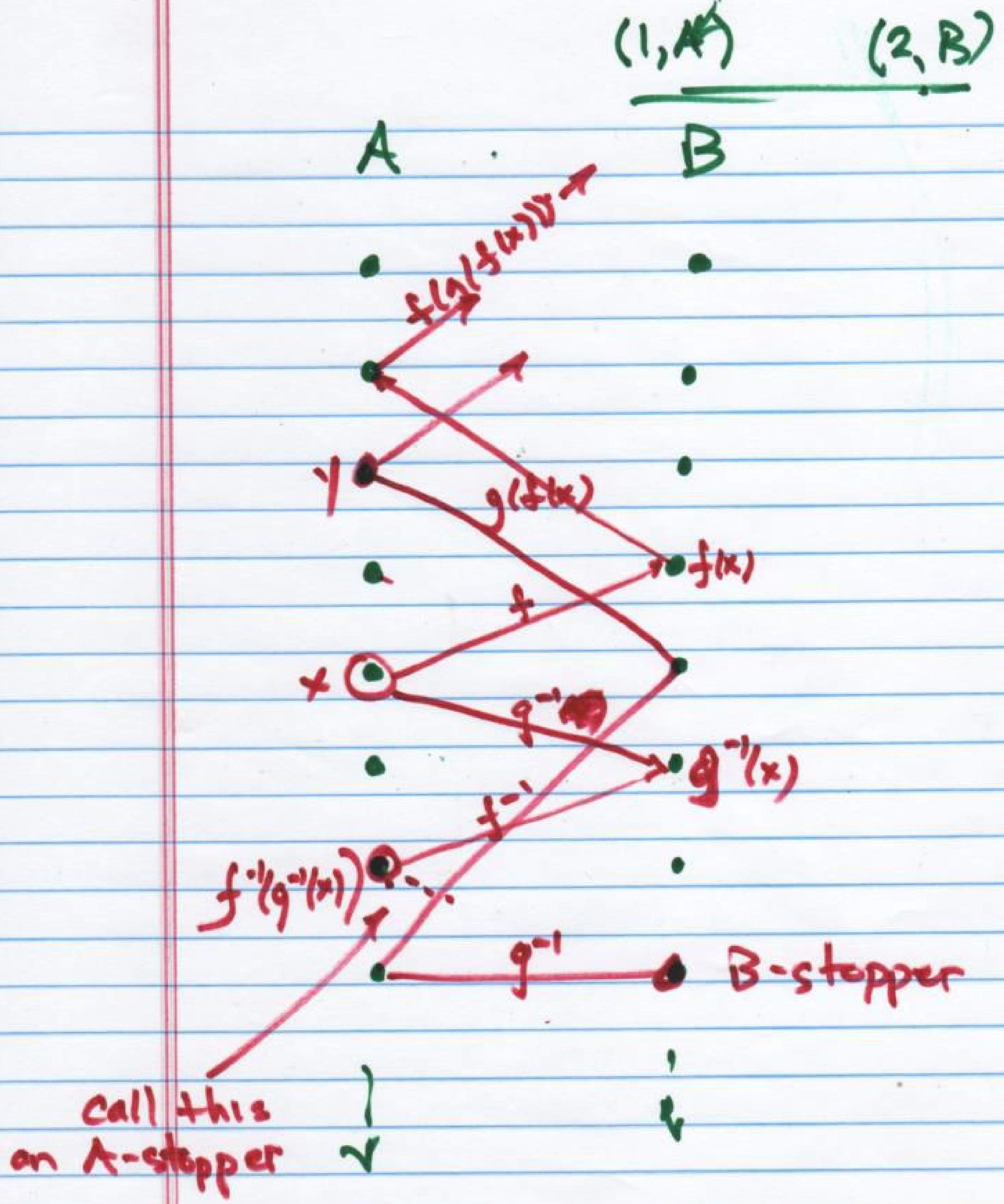
$\exists\, \phi : A \to B$ inj. & $\exists\, \psi : B - C$ inj.

So $\psi \circ \phi : A \to C$ injective, hence

$|A| \leq |C|$.

(iii) If $|A| \leq |B|$ & $|B| \leq |A|$ then....

$|A| = |B|$



(1, A)
(2, B)
A
B
f(g(f(x)))
y
g(f(x))
f(x)
f
x
g⁻¹
g⁻¹(x)
f⁻¹
f⁻¹(g⁻¹(x))
g⁻¹
B-stopper
call this an A-stopper

Cantor - Schröder - Bernstein (König)

If $|A| \leq |B|$ & $|B| \leq |A|$, then $|A| = |B|$

Pf: We are given $f: A \to B$ and $g: B \to A$ both injections.

We want $\phi: A \to B$.∍. ϕ is bijection.

Consider the fllwg. seq. for $x \in A$:

$\leftarrow \infty$

stops in A
stops in B

$$\begin{array}{ccccccccccc} \cdots f^{-1}(g^{-1}(x)) & \xleftarrow{f^{-1}} & g^{-1}(x) & \xleftarrow{g^{-1}} & x & \xrightarrow{f} & f(x) & \xrightarrow{g} & g(f(x)) & \xrightarrow{f} & \cdots \\ & & | & & | & & | & & | & & | \\ & & B & & A & & B & & A & & B \end{array}$$

$$\phi(x) = \begin{cases} f(x) & \text{if } x \in A\text{-stopper} \\ g^{-1}(x) & \text{if } g(x) \in B\text{-stopper} \\ f(x) & \text{all others} \end{cases}$$

A

König's Thm

Suppose $|A_i| < |B_i| \ \forall i \in \Delta$

Then $\left|\sum_i A_i\right| < \left|\prod_i B_i\right|$

Pf.. by inspection.

ND

Show card. multiplication distributes over card. addition. i.e.

Pf: Let. $\aleph_\alpha, \aleph_\beta, \aleph_\gamma$ be card. of sets.

(want to show) WTS: $\aleph_\alpha \times (\aleph_\beta + \aleph_\gamma) = \aleph_\alpha \times \aleph_\beta + \aleph_\alpha \times \aleph_\gamma$

WLOG: Let $\aleph_\alpha \geq \aleph_\beta \geq \aleph_\gamma$. Then,

Left-Hand Side: $\aleph_\alpha \times (\aleph_\beta + \aleph_\gamma) = \aleph_\alpha \times \aleph_\beta = \aleph_\alpha$

Right-Hand Side: $\aleph_\alpha \times \aleph_\beta + \aleph_\alpha \times \aleph_\gamma = \aleph_\alpha + \aleph_\alpha$

$= \aleph_\alpha$

$\aleph_\alpha = \aleph_\alpha$ ✓

//

⑤

WTS: $K_\alpha \cdot (K_\beta + K_\gamma) = K_\alpha \cdot K_\beta + K_\alpha \cdot K_\gamma$

Let $K_\alpha = |A|$ $K_\beta = |B|$ $K_\gamma = |C|$

LHS: $K_\beta + K_\gamma = |B \cup C|$

$K_\alpha \cdot (K_\beta + K_\gamma) = |\{(x,y) : x \in A, y \in B \cup C\}|$

RHS: $K_\alpha \cdot K_\beta = |\{(x,y) : x \in A, y \in B\}|$

$+$ $\qquad \cup$

LW: $K_\alpha \cdot K_\gamma = |\{(x,y) : x \in A, y \in C\}|$

$\downarrow$

$|\{(x,y) : x \in A, y \in B \cup C\}|$

SRI

6

Th^m/ $\exists \phi : \mathbb{R} \to \mathbb{R}^2 \ni \phi$ is bijective

Let $\alpha \in \mathbb{R}$ Restrict to $[0,1] \to [0,1]^2$

α can be expanded $0.d_1 d_2 d_3 d_4 \cdots d_n d_{n+1} \cdots$

$x = 0.d_1 d_3 d_5 \cdots$

$y : 0.d_2 d_4 d_6 \cdots$

Claim: $\alpha \mapsto (x,y)$ is bijective