Proposition 21: The following are true for relation R and sets $A, B \subseteq \text{dom}(R)$:

(i) $R(A \cup B) = R(A) \cup R(B)$

$\subseteq$: Let $y \in R(A \cup B)$. By definition, $\exists x \in A \cup B$ so that xRy. WLOG say $x \in A$, so $y \in R(A) \Rightarrow y \in R(A) \cup R(B)$.

$\supseteq$: Let $b \in R(A) \cup R(B)$. WLOG say $b \in R(A)$, so $\exists a \in A$, aRb. Since $a \in A$, we have $a \in A \cup B$ so $b \in R(A \cup B)$. □

(ii) $R(A \cap B) \subseteq R(A) \cap R(B)$

Let $y \in R(A \cap B)$, so $\exists x \in A \cap B$, xRy. Then $x \in A$ and $x \in B$, so $y \in R(A)$ and $y \in R(B)$; i.e., $y \in R(A) \cap R(B)$. □

(iii) $R(A \setminus B) \supseteq R(A) \setminus R(B)$

Let $c \in R(A) \setminus R(B)$, so $c \in R(A)$ and $c \notin R(B)$. Then $\exists a \in A$, aRc and $\forall b \in B$, $\neg bRc$. In particular, $a \notin B$. Then $a \in A \setminus B$, so $c \in R(A \setminus B)$. □

Proposition 25: If f is a function, the following are true:

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Follows by Proposition 21 (i). □

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$\subseteq$: Follows by Proposition 21 (ii).

$\supseteq$: Let $x \in f^{-1}(A \cap B)$. Then $f(x) \in A \cap B$, so $f(x) \in A \Rightarrow x \in f^{-1}(A)$ and $f(x) \in B \Rightarrow x \in f^{-1}(B)$. Thus $x \in f^{-1}(A) \cap f^{-1}(B)$. □

(iii) $f^{-1}(A \setminus B) = f^{-1}(A) - f^{-1}(B)$

$\subseteq$: Let $y \in f^{-1}(A \setminus B)$. Then $f(y) \in A \setminus B$, so $f(y) \in A$ and $f(y) \notin B$. Thus $y \in f^{-1}(A)$ and $y \notin f^{-1}(B)$, hence $y \in f^{-1}(A) \setminus f^{-1}(B)$.

$\supseteq$: Follows by Proposition 21 (iii).

□

①

(26) $S \subset 2^X$

Is there always an index set I .∋.

$\nu: I \to S$ is an indexing of the system

There is always $O \subset \Theta$ ← class of ordinals

ord

$1 \to r_1$
$2 \to r_2$

$\vdots$

$\Omega \to r_\Omega$

$\Omega + \omega$

Proof.

(27) $(\Rightarrow)$ Let $x \in \bigcup_{C \in S} \left(\bigcup_{a \in C} F_a \right)$. Then $\exists C \in S$,

and $\exists a \in C$ s.t. $x \in F_a$. Then $\forall a \in \bigcup C$,
~~$\exists x \in F_a$.~~ and then $a \in \bigcup S$.

Lenh

(28) $\bigcap_{a \in \cup S} F_a = \bigcap_{C \in S} \left(\bigcap_{a \in C} F_a \right)$.

($\Rightarrow$) Let $x \in \bigcap_{a \in \cup S} F_a$. Then $\forall C \in S$ s.t. and $\forall a \in C$, $x \in F_a$. But $x \in \bigcap_{C \in S} \left(\bigcap_{a \in C} F_a \right)$.

$\square$

(~~29~~) 31 MJ

$f : A \to B$ is surjective
$g : B \to C$ is surjective.

Show $g \circ f : A \to C$ is surjective.

pick $z \in C$, Then $\exists y \in B$ s.t.
$g(y) = z$ but $y \in B$
so $\exists x$ s.t. $f(x) = y$.

so $z = g(f(x))$

$f : A \to B$ injective
$g : B \to C$ injective

Show $g \circ f$ is injective.

$z_1, z_2 \in C$ and $z_1 \neq z_2$.
Then $\exists y_1, y_2 \in B$ s.t. $g(y_1) \neq g(y_2)$
Then $y_1 = y_2$ since g is injective.
So $f(x_1) = y_1 = y_2 = f(x_2)$ for $x_1, x_2 \in A$. thus $x_1 = x_2$

Proposition 28: $\bigcap_{a \in \cup S} F_a = \bigcap_{C \in S} \left(\bigcap_{a \in C} F_a \right)$

Proof.

($\Rightarrow$) Let $x \in \bigcap_{a \in \cup S} F_a$. Then $x \in F_a$, $\forall a \in \cup S$.

So $x \in F_a$, $\forall C \in S$, $\forall a \in C$.

Thus $x \in \bigcap_{C \in S} \left(\bigcap_{a \in C} F_a \right)$

($\Leftarrow$) Let $x \in \bigcap_{C \in S} \left(\bigcap_{a \in C} F_a \right)$.

Then $\forall C \in S$, and $\forall a \in C$, $x \in F_a$.

So $\forall a \in \cup S$ s.t. $x \in F_a$.

Thus $x \in \bigcap_{a \in \cup S} F_a$.

$\square$

Proposition 29: $f(\bigcup_{a\in A} F_a) = \bigcup_{a\in A} f(F_a)$

Proof.

$(\Rightarrow)$ Let $y \in f(\bigcup_{a\in A} F_a)$. Then $\exists x \in \bigcup_{a\in A} F_a$ s.t. $(x,y) \in f$.

Then $\exists a \in A$, $x \in F_a$, s.t. $x,y \in f$.

Then $\exists a \in A$ s.t. $y \in f(F_a)$

Thus $y \in \bigcup_{a\in A} f(F_a)$

$(\Leftarrow)$. Let $y \in \bigcup_{a\in A} f(F_a)$. Then $\exists a \in A$, $\exists y \in f(F_a)$.

Then $\exists x \in F_a$ s.t. $x,y \in f$.

Proposition 30: $f(\bigcap_{a\in A} F_a) \subseteq \bigcap_{a\in A} f(F_a)$

Let $y \in f[\bigcap_{a\in A} F_a]$. Then $\exists\, x \in \bigcap_{a\in A} F_a$ s.t. $(x,y) \in f$.

Then $\forall a \in A, x \in F_a$ s.t. $x,y \in f$.

Then $\forall a \in A, y \in f[F_a]$

Then $y \in \bigcap_{a\in A} f[F_a]$.