1-22

Filter on X:

Call $\hat{\mathcal{F}}$ a filterbase on X if:

$\hat{\mathcal{F}} \neq \emptyset$, $\emptyset \notin \hat{\mathcal{F}}$ and

$A, B \in \hat{\mathcal{F}} \Rightarrow A \cap B \in \hat{\mathcal{F}}$

A filterbase generates a filter by allowing all supersets.

So $\mathcal{F} = \langle \hat{\mathcal{F}} \rangle$

Then in $\mathcal{F}$,

$\emptyset \notin \mathcal{F}$, $\mathcal{F} \neq \emptyset$

$A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

and $A \subseteq B$, $A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$

Filterbases are not unique.

Ultrafilter - maximally refined.

Two types: free or principal

Consider $X = \mathbb{N}$ $\quad \mathcal{F} \subset 2^{\mathbb{N}}$

Claim $\{ S \subset \mathbb{N} \mid \text{for fixed } n \in \mathbb{N},\ n \in S \}$

is a <u>principal ultrafilter</u>.

Free ultrafilter $\mathcal{F}$ has property that

$$\bigcap_{S \in \mathcal{F}} \{ S \in \mathcal{F} \} = \emptyset$$

also called non-principal.

$X_i \quad I \in i$

$$\prod_{i \in \mathbb{N}} X_i$$

Let U be (free) ultrafilter of co-finite sets of $\mathbb{N}$.

Consider $\dfrac{\prod_{i \in \mathbb{N}} X_i}{U} \rightsquigarrow$ <u>ultrapower</u>

$X = \langle x_1, x_2, x_3 \ldots \ldots \rangle$

$Y = \langle y_1, y_2, y_3, \ldots \ldots \rangle$

if entries agree on indices given by $S \in U$

We say then $X \, [=]_U \, Y$

Now let $X_i = \mathbb{R} \;\; \forall i \in \mathbb{N}$

$[r]$ is equiv. class under U

$$\langle 1, 0, 2, 0, 3, 0, 4, 0, \ldots \rangle$$

$$\parallel$$

$$\langle 0, 0, 0, 0, \ldots \rangle$$

$$\parallel$$

$$\langle 1, 1, 1, 1, 1, 0, 2, 0, \ldots \rangle$$

hyperreal

$$\langle r, r, r, \ldots \rangle$$

$$\langle \ldots, r, r, r, \ldots \rangle$$

$$\langle \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots \tfrac{1}{2^n}, \ldots \rangle$$

$$\langle \tfrac{1}{2^2}, \tfrac{1}{4^2}, \tfrac{1}{8^2}, \ldots \tfrac{1}{2^{2n}} \ldots \rangle$$

"halo" of r is all infinitesimals

near r.

$$\langle 1, 2, 3, 4, 5, \ldots n \rangle$$

$$\langle \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \tfrac{1}{n} \rangle$$

All such numbers constitute

$^*\mathbb{R}$ (hyperreals) (1960)