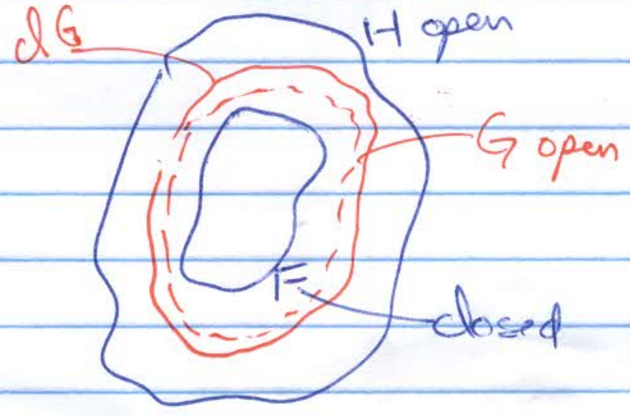
① You can follow this also in Munkres

Urysohn's Lemma:

If F_1, F_2 are closed and disjoint sets in a normal space X, then there exists a continuous function $f: X \to [0,1]$ with $f(F_1) = 0$ and $f(F_2) = 1$.

Lemma 1: Let X be normal. Then if $H \supset F$ with H open & F closed, $\exists\, G$ open such that $H \supset \bar{G} \supset G \supset F$



H^c is closed. $F \cap H^c = \phi$. By normality, $\exists$ open sets G, G^* such that $F \subset G$ and $H^c \subset G^*$. Moreover $G \cap G^* = \phi$.

But $G \cap G^* = \phi \Rightarrow G \subset (G^*)^c$, and

$H^c \subset G^* \Rightarrow (G^*)^c \subset H$

So $F \subset G \subset G^{*c} \subset H$

closed so $\bar{G} \subset G^{*c}$

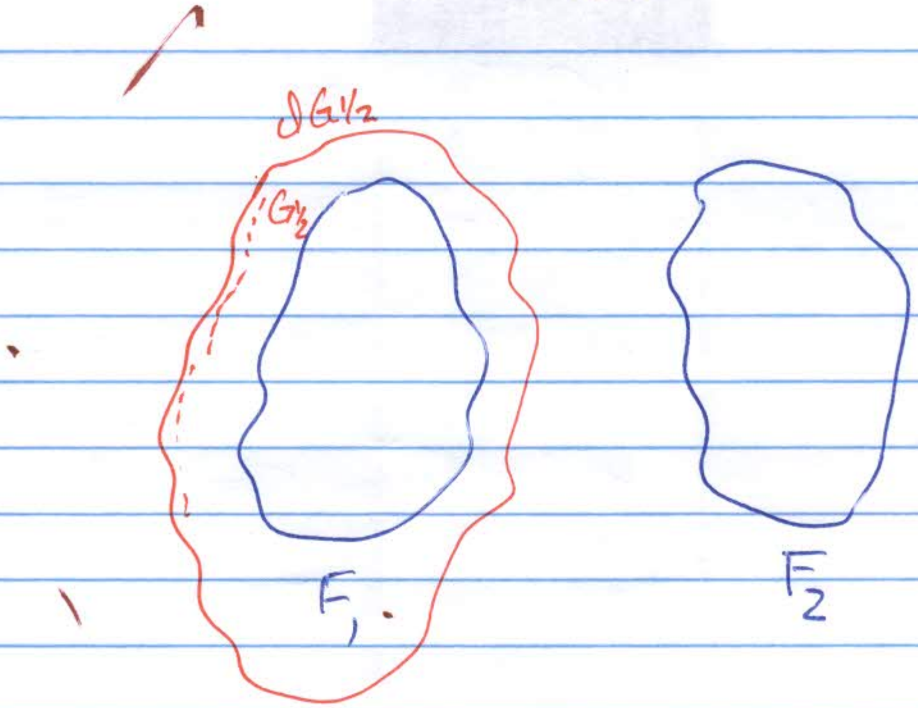
Hence $F \subset G \subset \mathrm{cl}\, G \subset H$. •

Lemma 2: $\mathrm{cl}\left\{\frac{k}{2^n} : k, n \in \mathbb{N}, k \leq 2^n\right\} \supset [0,1]$

UL: $F_1 \cap F_2 = \phi$ F_2^c open & $F_2^c \supset F_1$

So by Lemma 1, $\exists\, G_{1/2}$ $\ni$.

$$F_1 \subset G_{1/2} \subset \mathrm{cl}\, G_{1/2} \subset F_2^c$$



$G_{1/2}$ is open & contains F_1, F_2^c is open and contains $\partial G_{1/2}$ so by Lemma,

$$F_1 \subset G_{1/2} \subset \partial G_{1/2} \subset F_2^c$$

Now $G_{1/2}$ open $\supset F_1$ closed and

F_2^c open $\supset \partial G_{1/2}$ closed hence $\exists\, G_{1/4}$ & $G_{3/4}$

$$F_1 \subset G_{1/4} \subset \partial G_{1/4} \subset G_{1/2} \subset \partial G_{1/2} \subset G_{3/4} \subset \partial G_{3/4} \subset F_2^c$$

Iterate so that for each $t \in \{$set of dyadics$\}$,

$\exists\, G_t \ni$ if $t_1, t_2 \in D$, $t_1 < t_2$ then $\partial G_{t_1} \subset G_{t_2}$

Define $f: X \to [0,1]$ as follows:

$$f(x) = \begin{cases} \inf\{t : x \in G_t\} & \text{if } x \notin F_2 \\ 1 & \text{if } x \in F_2 \end{cases}$$

Certainly $0 \le f(x) \le 1$. Also

$F_1 \subset G_t \ \forall t$, so $f(F_1) = 0$.

By def'n, $f(F_2) = 1$

All that is left is continuity.

Need to show $f^{-1}\left[[0,a)\right]$ open

and $f^{-1}\left[(b,1]\right]$ open.

True if $f^{-1}\left[[0,a)\right] = \bigcup\{G_t : t < a\}$

$f^{-1}\left[(b,1]\right] = \bigcup\{\mathrm{cl}G_t^c : t > b\}$

⑤

$f^{-1}\left[[0,a)\right] = \bigcup_{t<a} G_t$ so open.

Let $x \in f^{-1}\left[[0,a)\right]$ then $f(x) \in [0,a)$, i.e

$0 \le f(x) < a$. D is dense in $[0,1]$, so

$\exists\, t_x \in D \cdot\ni\cdot f(x) < t_x < a$.

~~Then Since~~ then $f(x) = \inf\{t : x \in G_t\} < t_x < a$.

So $x \in G_{t_x}$ where $t_x < a$.

Now $x \in \bigcup\{G_t : t<a\}$. $\therefore$ Every element of $f^{-1}\left[[0,a)\right]$ belongs to $\bigcup\{G_t : t<a\}$.

i.e. $f^{-1}\left[[0,a)\right] \subseteq \bigcup\{G_t : t<a\}$.

Other way: Let $y \in \bigcup\{G_t : t<a\}$.

then $\exists\, t_y \in D \cdot\ni\cdot t_y < a$ and $y \in G_{t_y}$

$\therefore f(y) = \inf\{t : y \in G_t\} \le t_y < a$.

Hence y belongs to $f^{-1}\left[[0,a)\right]$